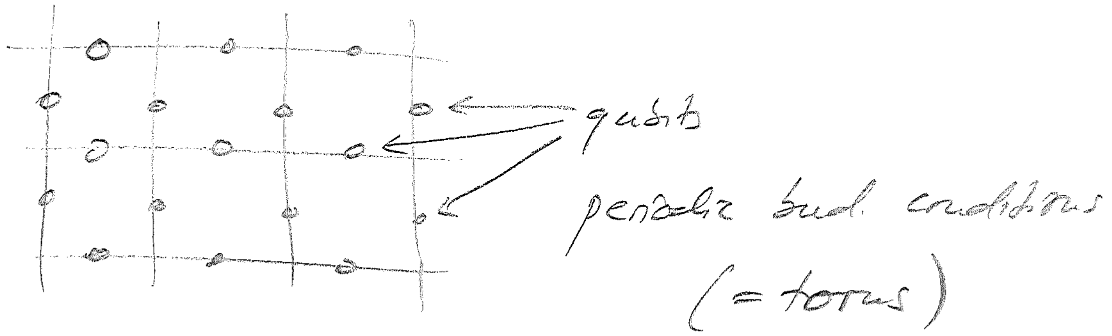
IX. Topological Quantum Memories & Quantum Computing

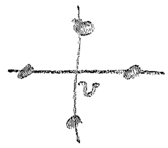
1) The Toric Code Model

Stabilizer on 2D lattice:

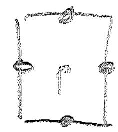


Stabilizers:

for each vertex v: $A_v = Z^{\otimes 4}$ on



for each plaquette p: $B_p = X^{\otimes 4}$ on



Note: $[A_v, A_{v'}] = [B_p, B_{p'}] = [A_v, B_p] = 0$

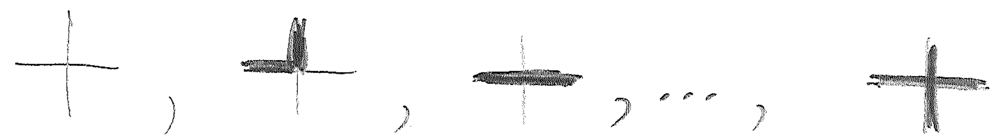
What is nature of +1 eigenstate(s) (stabilized subspace)?

Denote $|0\rangle$ as —— ("no line")

$|1\rangle$ as **——** ("line")

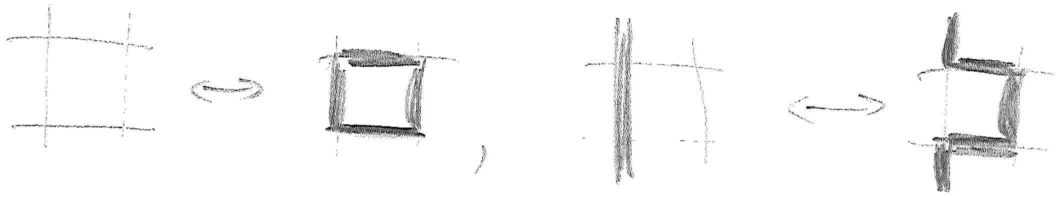
+1-state for A_v:

$\Rightarrow$ closed loop patterns:



+1-state for B_p:

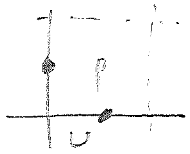
$\Rightarrow$ weight must be eq. for flipping <u>any plaquette</u>:



Possible joint +1-eigenstate:

Eq. wght. superposition of <u>all loop patterns</u> on torus

<u>Dimension of logical space?</u>



v/p pair: 2 qubits, 2 stabilizers

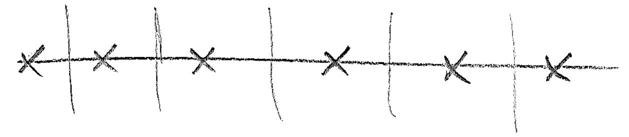
$\longrightarrow$ 1-dim. "code space"?

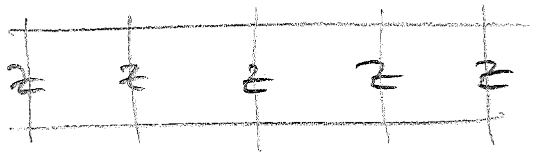
But: $\prod_v A_v = \mathbb{1}$ and $\prod_p B_p = \mathbb{1}$

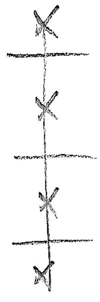
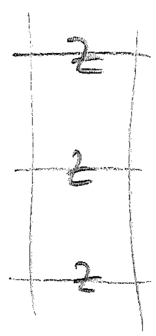
(and these are the only constraints!)

$\Rightarrow$ N qubits, N−2 constraints $\Rightarrow$ <u>2 logical qubits</u>!

<u>Logical operators</u>: ($\notin S$, but commute w/ S):
stabilizer

X_h =  around torus

Z_h = 

X_v =  , Z_v = 

- $Z_{h/v}$, $X_{h,v}$ comm. w. all A_v, B_p
- $[Z_h, X_h] = 0$; $[Z_v, X_v] = 0$, and

$$Z_h X_v = -X_v Z_h, \quad Z_v X_h = -X_h Z_v$$

$\Rightarrow$ (Z_h, X_v) & (Z_v, X_h) define logical qubits.

Are there <u>more local</u> logical operators?

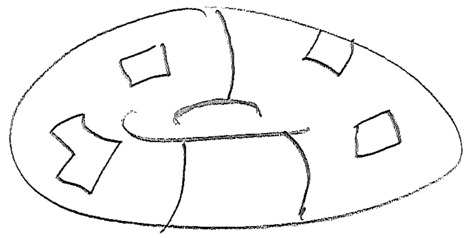
$\rightarrow$ No! Multiplication w/ A_v/B_p can deform loops, create local loops, or create an even # of loops around the torus, but cannot change the <u>parity</u> of loops around the torus.

⇒ <u>Logical operators</u> are topological!

<u>Interpretation</u> of code space in terms of <u>loop patterns</u>:

<u>+1/-1 eigenstate of Z_a</u>: even/odd # of loops cross Z_a

⇒ even/odd # of topologically non-trivial loops in vertical direction

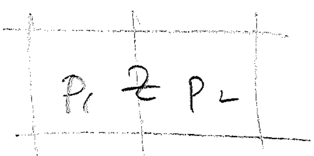


Same in horizontal direction.

⇒ 4 code states distinguished by global (topological) feature – locally states look identical!

<u>Effect of local errors?</u>

<u>Single Z</u>:

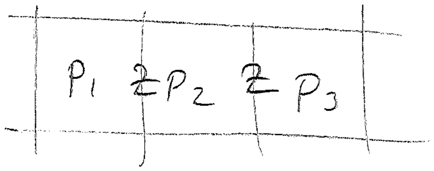


⇒ anticomm. w/ B_{p_1}, B_{p_2}

⇒ error syndrome on p_1 & p_2.

Can we "remove" error on e.g. B_{p_2}?

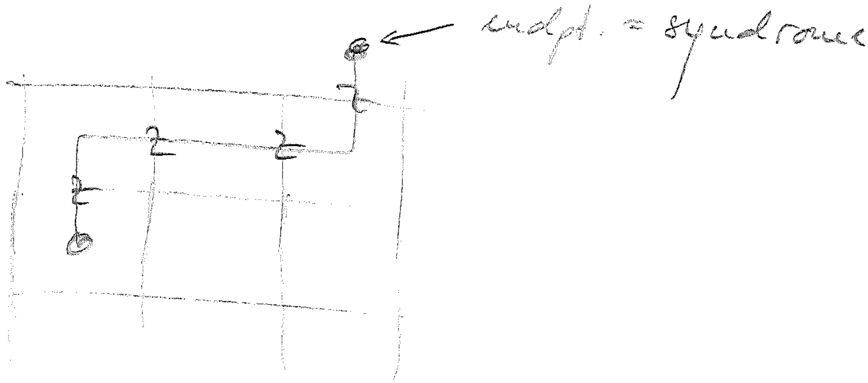
→ apply 2nd Z around B_{p_2}:



⟹ anticomm. w/ B_{p_1}, B_{p_2}

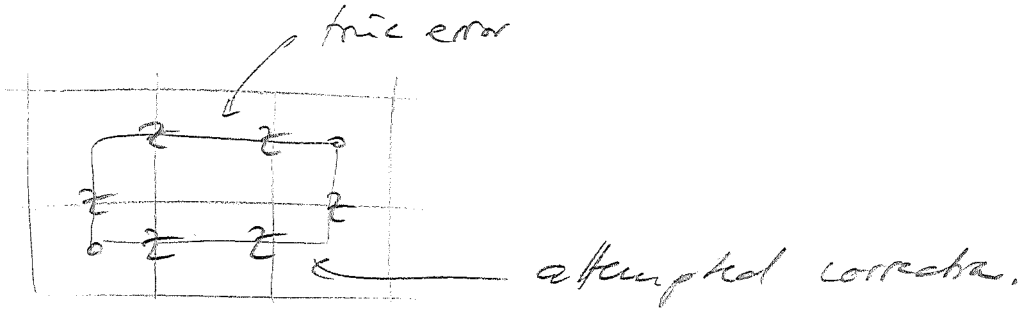
etc. ... ⟹ Z errors can form strings (on dual lattice)

w/ syndromes at the endpoints (= plaquettes)



Correction: Undo string betw. endpoints!

But: Syndrome does not reveal string (degen. code):



→ Error · Correction = topo. trivial loop $\in S$

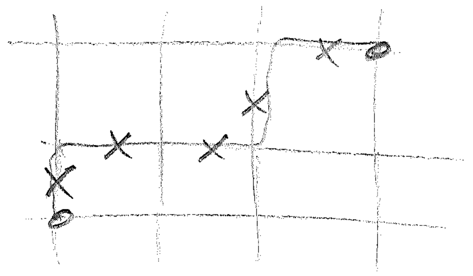
⟹ error corrected!

Possible source of errors: Pair of errors (= string) has propagated very far & we pair them up around the torus ⟹ Logical error!

⟹ Supressed exponentially in size of torus!

(Information topologically ~~protected~~: Encoded in global property, thus stable against local errors)

X errors: Similarly, but strings in original lattice w/ endpoints at vertices.



⟹ Topological quantum memory!

2) Toric Code as a topological model

Define 2D spin model (same lattice) as

$$H = -\sum_v A_v - \sum_p B_p$$

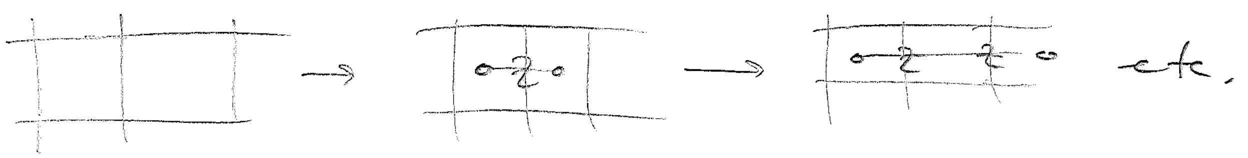
ground space $\equiv$ code space of TC.

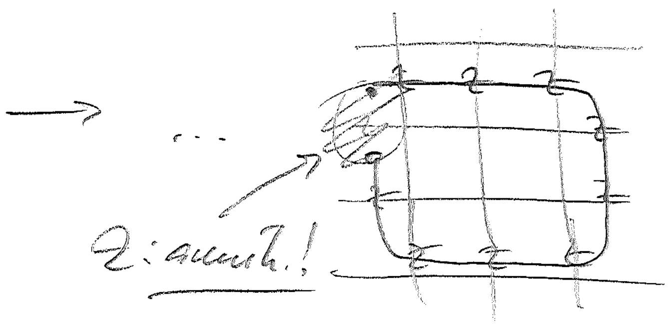
elementary excitations: $\equiv$ strings of Z's (or of X's) with 2 endpoints.

Excitations = endpoints. ($\equiv$ energy > 0)

$\Rightarrow$ excitations come in pairs!

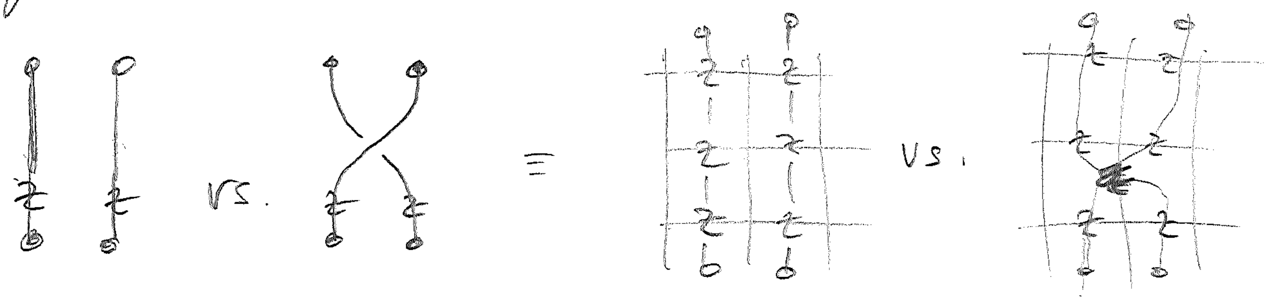
Creation / moving / annihilation of excitations by acting w Z (or X):





What is the statistics of excitations?

(1) Self-statistics:



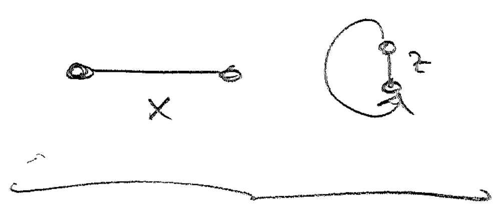
$\Rightarrow$ equal!

Same for X strings.

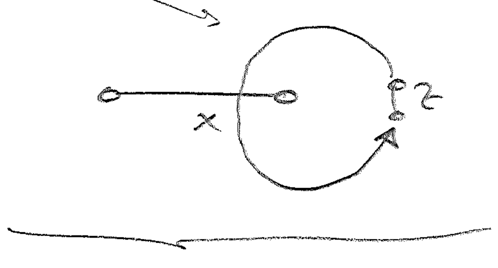
$\Rightarrow$ Bosonic self-statistics!

(2) Mutual statistics:

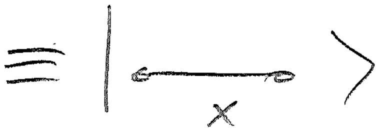
create Z string, move in circle, annihilate.



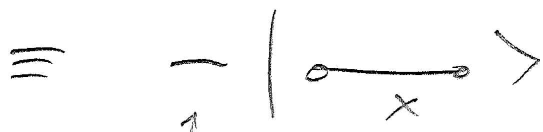
vs.



Z loop vanishes;



Z loop can be moved through X string and then vanishes:



from $XZ = -ZX$.

$\Rightarrow$ fermionic mutual statistics!

(3) ⇒ Combined particle of one X and one Z string has fermionic statistics!

Emergent fermions in a spin ($\hat{=}$ bosonic) model!

More complex spin models:

Can have anyonic excitations
(with arbitrary braiding phases $e^{i\theta}$)
and even non-abelian anyons
(with statistics described by matrices).

Similar in Fractional Quantum Hall Effect:

Emergent quasi-particles w/ anyonic statistics
(by combining electrons + flux quanta).

Can be used for q. computation: Braid excitation & compute using the non-abelian braiding "phases"
(can be read out interferometrically)

Q. Information stored in global state ⇒ topologically protected.