Exercise Sheet 1

Quantum Information

To be returned no later than April 23, 2015

(25 points) **Problem 1: Pauli matrices.** The following matrices are called *Pauli matrices*:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

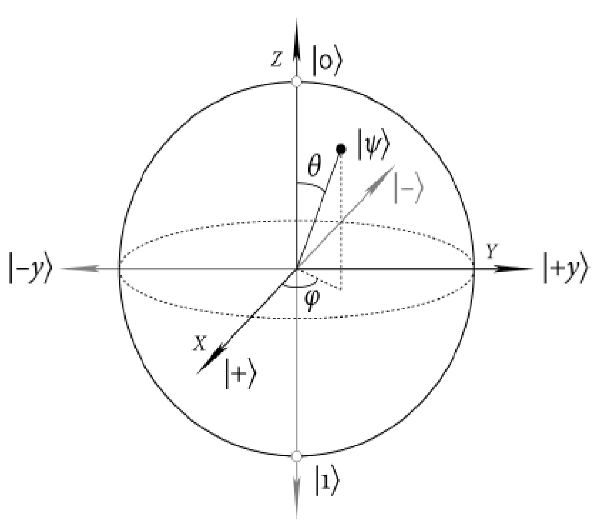
They are written in $|0\rangle$, $|1\rangle$ basis, which is the eigenbasis for Z matrix. This basis is called computational basis.

1) Show that the Pauli matrices are all Hermitian, unitary and they square to the identity.
2) Show that the Pauli matrices either commute or anticommute.
3) Let us label the Pauli matrices as $\sigma_0 = \mathbb{I}$, $\sigma_1 = X$, $\sigma_2 = Y$ and $\sigma_3 = Z$. Show that $\mathrm{Tr}(\sigma_i \sigma_j) = 2\delta_{ij}$ for all $i, j \in \{0, ..., 3\}$. Here δ_{ij} is a Kronecker delta function.
4) Write each operator X, Y and Z using bra-ket notation with states takes from a computation basis.
5) Find the eigenstates, eigenvalues, and diagonal representation of X, Y and Z matrices.
6) Determine the set of measurement operators corresponding to a measurement of Y observable.
7) For a state $|\psi\rangle$ write the possible states it can collapse to after the measurement of Y observable and find the corresponding probabilities when it can happen.
8) Write all tensor products of Pauli matrices as 4×4 matrices.

(10 points) **Problem 2: Bloch sphere for pure states.** Any pure qubit state $|\psi\rangle$ can be written in the following way

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle\,,$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$ are Bloch sphere angles.



Determine Bloch sphere angles θ and ϕ for:
1) the eigenstates of the Pauli X operator.
2) the eigenstates of the Pauli Y operator.
3) the eigenstates of the Pauli Z operator.
And **show** that each pair lie on the same axis.

(20 points) **Problem 3: Bloch sphere for all states.** 1) Prove that any single-qubit mixed state ρ can be written in the following way

$$\rho = \frac{1}{2}(\mathbb{I} + \boldsymbol{r} \cdot \boldsymbol{\sigma}) = \frac{1}{2}(\mathbb{I} + r_x X + r_y Y + r_z Z),$$

where $\boldsymbol{\sigma} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ is a vector consisting of Pauli matrices and $\boldsymbol{r} \in \mathbb{R}^3$ is called the *Bloch vector* of the system.

2) Show that for density operators $|\boldsymbol{r}| \leq 1$.

3) Prove that for pure states $|\boldsymbol{r}| = 1$. Hint: First prove that a state ρ is pure if and only if $\rho^2 = \rho$. These results mean that the surface of the Bloch sphere represents all pure states and the interior corresponds to all mixed states.

4) Give the Bloch vectors corresponding to the following states, and draw then on the Bloh sphere:

1) $\frac{1}{2}\mathbb{I} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$;

2) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$;

3) $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

(15 points) **Problem 4: No-cloning theorem.** Suppose that two states $|\psi\rangle$ and $|\psi^\perp\rangle$ are orthogonal: $\langle\psi|\psi^\perp\rangle = 0$. Construct a two-qubit unitary that can copy the states, i.e. find a unitary U that acts as follows:

$$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$$
$$U|\psi^\perp\rangle|0\rangle = |\psi^\perp\rangle|\psi^\perp\rangle.$$

(30 points) **Problem 5: Tensor product.**

The following states are known as *Bell states* and are the most important entangled states for a two-qubit system AB. These states are orthogonal and form a *Bell basis*:

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(|00\rangle_{AB} + |11\rangle_{AB}\Big),$$

$$|\Phi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(|00\rangle_{AB} - |11\rangle_{AB}\Big),$$

$$|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(|01\rangle_{AB} + |10\rangle_{AB}\Big),$$

$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(|01\rangle_{AB} - |10\rangle_{AB}\Big).$$

Suppose that Alice possesses a qubit $|\psi\rangle_{A'} = \alpha|0\rangle_{A'} + \beta|1\rangle_{A'}$. Suppose also that she shares a maximally entangled state $|\Phi^+\rangle_{AB}$ with Bob.
1) Write explicitly the joint state of systems A', A and B.
2) Rewrite the joint system $A'A$ in the Bell basis, i.e. the only states allowed on $A'A$ system are Bell states. In other words, show that the initial state can be written as

$$\begin{aligned}|\psi\rangle_{A'}\left|\Phi^+\right\rangle_{AB} =& \frac{1}{2}\Big(\left|\Phi^+\right\rangle_{A'A}(\alpha|0\rangle_B + \beta|1\rangle_B) + \left|\Phi^-\right\rangle_{A'A}(\alpha|0\rangle_B - \beta|1\rangle_B) \\ &+ \left|\Psi^+\right\rangle_{A'A}(\alpha|1\rangle_B + \beta|0\rangle_B) + \left|\Psi^-\right\rangle_{A'A}(\alpha|1\rangle_B - \beta|0\rangle_B)\Big).\end{aligned}$$

3) For a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ calculate the expressions: $X|\psi\rangle, Z|\psi\rangle$ and $XZ|\psi\rangle$.
4) Simplify the state you have obtained in part 2) using the expressions from part 3).
5) Suppose now that Alice performs a Bell measurement on her systems $A'A$, i.e. she performs a measurement with the following set of measuring operators

$$\{\left|\Phi^+\right\rangle\left\langle\Phi^+\right|_{A'A}, \left|\Phi^-\right\rangle\left\langle\Phi^-\right|_{A'A}, \left|\Psi^+\right\rangle\left\langle\Psi^+\right|_{A'A}, \left|\Psi^-\right\rangle\left\langle\Psi^-\right|_{A'A}\}.$$

Write the states to which the initial state collapses onto and calculate the corresponding probabilities.