Exercise Sheet 2

Quantum Information

To be returned no later than April 30, 2015

(20 points) **Problem 1: Purification.**
1) Find a purification of the following classical-quantum state:

$$\rho_{XA} = \sum_j p_j \left|j\right\rangle \left\langle j\right|_X \otimes \rho_j^A,$$

where $\{\left|j\right\rangle_X\}_j$ is an orthonormal basis of system X, $0 \leq p_j \leq 1$ are probabilities, i.e. $\sum_j p_j = 1$, and ρ_j^A is a density matrix on system A for every j.
2) Let $\{p_j, \rho_j^A\}_j$ be an ensemble of density operators. Suppose that $\left|\psi_j\right\rangle_{AE}$ is a purification of ρ_j^A. The expected density operator of the ensemble is

$$\rho_A = \sum_j p_j \rho_j^A.$$

Find a purification of ρ_A.
Hint: Remember that there is no restriction on a purifying system. You may choose it as big/small as you want.

(20 points) **Problem 2: Linearity of a superoperator.**
Consider a non-linear super operator

$$\mathcal{E}(\rho) = \exp\{i\pi Z \operatorname{Tr}(Z\rho)\}\, \rho\, \exp\{-i\pi Z \operatorname{Tr}(Z\rho)\},$$

where Z is a Pauli Z operator.
1) Check that $\mathcal{E}$ is positive and trace-preserving.

Suppose that the initial density matrix is $\rho = \frac{1}{2}\mathbb{I}$, realized as the ensemble

$$\rho = \frac{1}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1}{2}\left|1\right\rangle\left\langle 1\right|,$$

where $\left|0\right\rangle$ and $\left|1\right\rangle$ are eigenvalues of Z operator.
2) Find the evolution of this operator governed by $\mathcal{E}$, i.e. calculate $\mathcal{E}(\rho)$.

Now suppose that, immediately after preparing the ensemble ρ above, we do nothing if the state has been prepared as $\left|0\right\rangle$, but we rotate it to $\left|+\right\rangle$ if it has been prepared as $\left|1\right\rangle$. The density matrix is now

$$\rho' = \frac{1}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1}{2}\left|+\right\rangle\left\langle +\right|.$$

3) Find the evolution of this operator governed by $\mathcal{E}$, i.e. calculate $\mathcal{E}(\rho')$. What happens to a state if it was prepared as $|0\rangle$?

Now compare the results obtained in parts 2) and 3). If the calculations were carried out correctly, you will find that a state $|0\rangle$ evolves differently under the same evolution in these two scenarios. But what is the difference between the two cases? The difference was that if the spin was initially prepared as $|1\rangle$, we took different actions: doing nothing in one case, but rotating the spin in the second case. Yet we have found that the spin behaves differently in the two cases, even if it was initially prepared as $|0\rangle$!

(15 points) **Problem 3: Complete positivity.** Show that the evolution with Kraus operators M_α given by

$$\mathcal{E}(\rho) = \sum_\alpha M_\alpha \rho M_\alpha^\dagger$$

is *completely positive.*
Hint: Remind that a completely positive map is such a map that the output of the tensor product $I^k \otimes \mathcal{E}$ for any finite k is a positive operator whenever the input is a positive operator (the input operator now lives on a tensor-product Hilbert space).

(15 points) **Problem 4: Measurement.** Suppose that the initial state of system AB is

$$|\phi_\lambda\rangle = \sqrt{\lambda}\,|00\rangle + \sqrt{1-\lambda}\,|11\rangle\,.$$

The goal is to obtain a maximally entangled state $\left|\phi_{1/2}\right\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ with some probability after a measurement on system A.
1) Show that the following operators satisfy the completeness condition: $\Pi_0 = (|0\rangle\langle 0| + \sqrt{\gamma}\,|1\rangle\langle 1|)_A \otimes \mathbb{I}_B$ and $\Pi_1 = \sqrt{1-\gamma}\,|1\rangle\langle 1|_A \otimes \mathbb{I}_B$, with $0 \le \gamma \le 1$. In other words, show that $\sum_j \Pi_j^* \Pi_j = \mathbb{I}_{AB}$.
2) Find possible states to which the initial one collapses onto and calculate the corresponding probabilities.
3) Find a value γ such that one of post-measurement states becomes a maximally entangled state. Calculate the corresponding probability with which the initial state becomes a maximally entangled state.
Note that if $\lambda = \frac{1}{2}$ initially, probability of the initial state collapsing onto the maximally entangled one is 1.

(30 points) **Problem 5: Channels.** In this problem, we will study some commonly appearing quantum channels. In addition to the problems listed, verify for each channel that it is a CPTP map (completely positive trace preserving map) and give its Kraus representation.
1) *Dephasing channel.* This channel acts as follows on any given density operator:

$$\rho \to (1-p)\rho + pZ\,\rho\,Z.$$

Show that the action of the dephasing channel on the Bloch vector is

$$\frac{1}{2}(\mathbb{I} + r_x X + r_y Y + r_z Z) \to \frac{1}{2}(\mathbb{I} + (1-2p) r_x X + (1-2p) r_y Y + r_z Z),$$

so that the channel preserves any component of the Bloch vector in the Z direction, while shrinking any component in the X or Y direction.

2) *Amplitude damping channel.* Let the parameter γ denote the probability of decay so that $0 \leq \gamma \leq 1$. The channel with transmission parameter $1-\gamma$ is specified by two Kraus operators:

$$\Pi_0 = \sqrt{\gamma}\,|0\rangle\langle 1|\,, \quad \Pi_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}\,|1\rangle\langle 1|\,.$$

a) Consider a single-qubit density operator with the following matrix representation with respect to the computation basis

$$\rho = \begin{pmatrix} 1-p & \eta \\ \eta^* & p \end{pmatrix},$$

where $0 \leq p \leq 1$ and η is some complex number. Find the matrix representation of this density operator after the action of the amplitude damping channel.

b) Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel $\mathcal{N}_1$ with transmission parameter $1-\gamma_1$ and consider another amplitude damping channel $\mathcal{N}_2$ with transmission parameter $1-\gamma_2$. Show that the composition channel $\mathcal{N}_1 \circ \mathcal{N}_2$ is an amplitude damping channel with transmission parameter $(1-\gamma_1)(1-\gamma_2)$.

3) *Twirling operation.* Show that randomly applying the Pauli operators $\mathbb{I}, X, Y, Z$ with uniform probability to any density operator gives the maximally mixed state:

$$\frac{1}{4}\rho + \frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z = \frac{1}{2}\mathbb{I}.$$

Hint: Represent the density operator as $\rho = \frac{1}{2}(\mathbb{I} + r_x X + r_y Y + r_z Z)$ and apply the commutation rules of the Pauli operators.

4) *Classical-quantum channel.* A classical-quantum channel is one that first measures the input state in a particular orthonormal basis and outputs a density operator conditional on the result of the measurement:

$$\mathcal{E}(\rho) = \sum_k \langle k|\,\rho\,|k\rangle\,\sigma_k,$$

where $\{|k\rangle\}_k$ is an orthonormal basis for the Hilbert space on which the initial density operator ρ acts, and $\{\sigma_k\}_k$ is a set of density operators. Show that the classical-quantum channel is an *entanglement-breaking channel*– i.e., if we input the B system of an entangled state ψ_{AB}, then the resulting state on AB is no longer entangled.

Hint: Remind that a separable state can always be written as a convex combination of pure product states

$$\sum_j p_j\,|\phi_j\rangle\langle\phi_j|_A \otimes |\psi_j\rangle\langle\psi_j|_B\,.$$