Exercise Sheet 3

Quantum Information

To be returned no later than May 07, 2015

(20 points) **Problem 1: Cirel'son's inequality.**
This inequality provides the largest possible quantum mechanical violation of the CHSH inequality.
Suppose that a, a', b, b' are Hermitian operators with eigenvalues ± 1, so that

$$a^2 = a'^2 = b^2 = b'^2 = \mathbb{I},$$

and suppose that "Alice's observables" a, a' commute with "Bob's observables" b, b'

$$0 = [a, b] = [a, b'] = [a', b] = [a', b'].$$

Define

$$C = ab + a'b + ab' - a'b'.$$

1) Evaluate C^2.

The norm of a bounded operator M is defined by

$$\|M\| = \sup_{|\psi\rangle} \frac{\| M |\psi\rangle \|}{\| \, |\psi\rangle \, \|},$$

that is the norm of M is the maximum eigenvalue of $\sqrt{M^\dagger M}$.
2) Verify that the norm has the properties

$$\|MN\| \leq \|M\|\|N\|,$$

$$\|M + N\| \leq \|M\| + \|N\|.$$

3) Find the upper bound on the norm $\|C^2\|$.
4) Notice that for Hermitian operators $\|C^2\| = \|C\|^2$. Derive the the upper bound on the norm $\|C\|$.
This inequality (if done correctly) is known as Cirel'son's inequality.

(20 points) **Problem 2: Remote state preparation.**
Remote state preparation is a variation on the teleportation protocol. We consider a simple example of a remote state preparation protocol. Suppose Alice possesses a classical description of a state $|\psi\rangle = (|0\rangle + e^{i\phi} |1\rangle)/\sqrt{2}$ (on the equator of the Bloch sphere) and she shares an ebit $|\Phi^+\rangle$ with Bob. Alice would like to prepare this state on Bob's system. Show that Alice can prepare this state on

Bob's system if she measures her system A in the $(\psi, \psi^\perp)$ basis, transmits one classical bit, and Bob performs a recovery operation conditional on the classical information, but independent of state $|\psi\rangle$.

(20 points) **Problem 3: Third-party controlled teleportation.**
Third-party controlled teleportation is another variation on the teleportation protocol. Suppose that Alice, Bob, and Charlie possess a GHZ state:

$$|\Phi_{GHZ}\rangle = \frac{|000\rangle_{ABC} + |111\rangle_{ABC}}{\sqrt{2}}$$

Alice would like to teleport an arbitrary qubit to Bob. She performs the usual steps in the teleportation protocol. Give the final steps that Charlie should perform and the information that he should transmit to Bob in order to complete the teleportation protocol.
Hint: The resource inequality for the protocol is as follows:

$$[qqq]_{ABC} + 2[c \to c]_{A \to B} + [c \to c]_{C \to B} \geq [q \to q]_{A \to B},$$

where $[qqq]_{ABC}$ represents the resource of the GHZ state shared between Alice, Bob, and Charlie, and the other resources are as before with the directionality of communication indicated by the corresponding subscript.

(20 points) **Problem 4: Dephasing channel and teleportation.**
Show that it is possible to simulate a dephasing qubit channel by the following technique. First, Alice prepares a maximally entangled Bell state $|\Phi^+\rangle$. She sends half of it to Bob through a dephasing qubit channel. She and Bob perform the usual teleportation protocol. Show that this procedure gives the same result as sending a qubit through a dephasing channel.
Hint: This result holds because the dephasing channel commutes with all Pauli operators.

(20 points) **Problem 5: Gate teleportation.**
Gate teleportation is yet another variation of quantum teleportation that is useful in fault-tolerant quantum computation. Suppose that Alice would like to perform a single-qubit gate U on a qubit in state $|\psi\rangle$. Suppose that the gate U is difficult to perform, but that $U\sigma_j U^\dagger$, where σ_j is one of the single-qubit Pauli operators, is much less difficult to perform. A protocol for gate teleportation is as follows. Alice and Bob first prepare the ebit $U_B |\Phi^+\rangle_{AB}$. Alice performs a Bell measurement on her qubit $|\psi\rangle_{A'}$ and system A. She transmits two classical bits to Bob and Bob performs one of the four corrective operations $U\sigma_j U^\dagger$ on his qubit. Show that this protocol works, i.e., Bob's final state is $U |\psi\rangle$.