Exercise Sheet 4

Quantum Information

To be returned no later than May 15, 2015

(20 points) **Problem 1: Majorization.**
1) Show that $x \prec y$ if and only if for all real t, $\sum_{j=1}^{d} \max\{x_j - t, 0\} \leq \sum_{j=1}^{d} \max\{y_j - t, 0\}$, and $\sum_{j=1}^{d} x_j = \sum_{j=1}^{d} y_j$.
2) Use the previous problem to show that the set of x such that $x \prec y$ is convex.
3) Give an example of two vectors x and y that are not related by majorization.
4) Let f be a convex function, and define $F(x) = \sum f(x)$. Show that $x \prec y$ implies that $F(x) \leq F(y)$ (a property known as "Schur convexity").
5) Use the previous problem to show that the entanglement $E(|\psi\rangle) = S(\mathrm{tr}_A|\psi\rangle\langle\psi|)$ cannot be increased by LOCC.

(25 points) **Problem 2: Equivalence of definitions.**
1) Prove that $x \prec y$ if and only if $x = \sum_j p_j P_j y$ for some probability distribution p_j and permutation matrices P_j.
Hint: Suppose that $x \prec y$. Use induction on the dimension d and suppose that x and y are $d+1$-dimensional vectors. The goal is to find a combination of permutations D such that a d-dimensional vector y' created by eliminating a first entry from the vector Dy majorizes vector $x' = (x_2, \ldots, x_{d+1})$.
-Show that there exist j and $T \in [0, 1]$ such that $x_1 = ty_1 + (1-t)y_j$.
-Define $D = tI + (1-t)T$, where T is the permutation matrix which transposes the 1st and j-th matrix elements. Write the components of vector Dy.
-Define x' and y' by eliminating the first entry from x and Dy respectively. Show that $x' \prec y'$.
-Invoke the inductive hypothesis and finish the proof in this direction.
To prove the other direction, you may use Problem 1 above.
2) Show that $x \prec y$ if and only if $x = Dy$ for some doubly stochastic D.
Hint: You may use Birkhoff's theorem without proof, which states that any doubly stochastic D can be written as $D = \sum p_j P_j$ with p_j a probability distribution and P_j permutations.

(20 points) **Problem 3: LOCC transformations.**
Suppose $|\psi\rangle$ can be transformed to $|\phi\rangle$ by LOCC. A general LOCC protocol can involve an arbitrary number of rounds of measurement and classical communication. In this problem, we will show that any LOCC protocol can be realized in a single round with only one-way communication, i.e., a protocol involving just the following steps: Alice performs a single measurement described by measurement operators K_j , sends the result j to Bob, who performs a unitary operation U_j on his system.

The idea is simply to show that the effect of any measurement Bob can do may be simulated by Alice (with one small caveat) so all Bob's actions can actually be replaced by actions by Alice!
1) First, suppose that Bob performs a measurement with operators $M_j = \sum_{kl} M_{j,kl} \left|k\right\rangle_B \left\langle l\right|_B$ on a pure state $\left|\psi\right\rangle_{AB} = \sum \lambda_l \left|l\right\rangle_A \left|l\right\rangle_B$, with resulting state denoted as $\left|\psi_j\right\rangle$. Now suppose that Alice performs a measurement with operators $N_j = \sum_{kl} M_{j,kl} \left|k\right\rangle_A \left\langle l\right|_A$ on a pure state $\left|\psi\right\rangle$, with resulting state denoted as $\left|\phi_j\right\rangle$. Show that there exist unitaries U_j on system A and V_j on system B such that $\left|\psi_j\right\rangle = (U_j \otimes V_j) \left|\phi_j\right\rangle$.
2) Explain how a multi-round protocol can now be done with one measurement done by Alice and one unitary operation done by Bob conditioned on Alice's outcome.

(20 points) **Problem 4: Entanglement catalysis.**
Suppose Alice and Bob share a pair of four level systems in the state $\left|\psi\right\rangle = \sqrt{0.4}\left|00\right\rangle + \sqrt{0.4}\left|11\right\rangle + \sqrt{0.1}\left|22\right\rangle + \sqrt{0.1}\left|33\right\rangle$. Show that it is not possible for them to convert this state by LOCC to the state $\left|\phi\right\rangle = \sqrt{0.5}\left|00\right\rangle + \sqrt{0.25}\left|11\right\rangle + \sqrt{0.25}\left|22\right\rangle$. Imagine, however, that a friendly bank is willing to offer them the loan of a catalyst, an entangled pair of qubits in the state $\left|c\right\rangle = \sqrt{0.6}\left|00\right\rangle + \sqrt{0.4}\left|11\right\rangle$. Show that it is possible for Alice and Bob to convert the state $\left|\psi\right\rangle \otimes \left|c\right\rangle$ to $\left|\phi\right\rangle \otimes \left|c\right\rangle$ by local operations and classical communication, allowing them to return the catalyst $\left|c\right\rangle$ to the bank after the transformation is complete.

(15 points) **Problem 5: Fidelity.**
1) Let a be a hermitian operator X, and let λ_i denote its eigenvalues. Then, the trace norm $\|X\|_1 \equiv \|X\|_{\mathrm{tr}}$ is defined as $\|X\|_1 := \sum_i |\lambda_i|$, and the operator norm $\|X\| \equiv \|X\|_\infty$ as $\|X\|_\infty = \max_i |\lambda_i|$ (cf. Sheet 3, Problem 1). Show that for A, B hermitian,

$$|\mathrm{tr}(AB)| \leq \|A\|_1\| \, \|B\|_\infty \, .$$

2) Use the above inequality to derive the bound

$$\Big| \left\langle\psi\right| O \left|\psi\right\rangle - \left\langle\phi\right| O \left|\phi\right\rangle \Big| \leq 2\sqrt{1 - |\left\langle\psi|\phi\right\rangle|^2} \, \|O\|_\infty \, .$$

(*Note:* The above inequality still holds for general operators, where the trace norm is defined as $\|X\|_1 = \mathrm{tr}|X|$, where $|X| := \sqrt{X^\dagger X}$, and the operator norm as $\|X\|_\infty := \lambda_{\max}(|X|)$. It is a special case of the corresponding Hölder inequality $|\mathrm{tr}(AB^\dagger)| \leq \|A\|_p \|B\|_q$ with $1/p + 1/q = 1$, where $\|X\|_p := \big(\mathrm{tr}|X|^p\big)^{1/p}$.)