Exercise #06

① Negativity

1 $N(\rho) = \frac{1}{2}(\|\rho^{T_A}\|_1 - 1)$

$$N\left(\sum_j p_j \rho_j\right) = \frac{1}{2}\left(\left\| \sum_j p_j \rho_j^{T_A}\right\|_1 - 1\right)$$

$$\overset{\Delta \text{ ineq.}}{\leq} \frac{1}{2}\left(\sum_j p_j \|\rho_j^{T_A}\|_1 - \sum_j p_j\right)$$

$$= \sum_j p_j \frac{1}{2}\left(\|\rho_j^{T_A}\|_1 - 1\right)$$

$$= \sum_j p_j N(\rho_j)$$

$\Rightarrow N(\rho)$ is convex.

2. $\|\rho_1 \otimes \rho_2\|_1 = \sum_{i,j} |\alpha_i \beta_j| = \sum_{i,j} |\alpha_i||\beta_j| = \sum_i |\alpha_i| \sum_j |\beta_j|$

$$= \|\rho_1\|_1 \cdot \|\rho_2\|_2$$

$(\rho_1 \otimes \rho_2)^{T_A} = \rho_1^{T_A} \otimes \rho_2^{T_A}$

$E_N(\rho) = \log_2 \|\rho^{T_A}\|_1$

$E_N(\rho_1 \otimes \rho_2) = \log_2 \|(\rho_1 \otimes \rho_2)^{T_A}\|_1 = \log_2 \|\rho_1^{T_A} \otimes \rho_2^{T_A}\|_1$

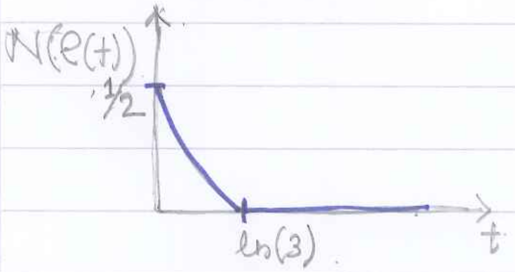
$$= \log_2(\|\rho_1^{T_A}\|_1 \cdot \|\rho_2^{T_A}\|) = \log_2 \|\rho_1^{T_A}\|_1 + \log_2 \|\rho_2^{T_A}\|_1$$

$\Rightarrow E_N(\rho_1 \otimes \rho_2) = E_N(\rho_1) + E_N(\rho_2)$

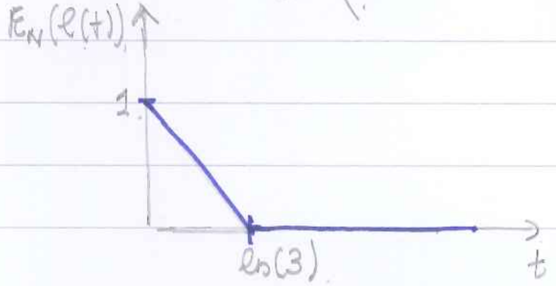
③ $$\rho^{T_A}(t) = \begin{pmatrix} P_+ & 0 & 0 & 0 \\ 0 & P_- & \frac{1}{2}e^{-t} & 0 \\ 0 & \frac{1}{2}e^{-t} & P_- & 0 \\ 0 & 0 & 0 & P_+ \end{pmatrix}$$

$\|\rho^{T_A}(t)\|_1 = \frac{3}{4}|(1+e^{-t})| + |\frac{1}{4} - \frac{3}{4}e^{-t}|$

$$N(\rho(t)) = \tfrac{1}{2}\left(\tfrac{3}{4}|1+e^{-t}| + |\tfrac{1}{4} - \tfrac{3}{4}e^{-t}| - 1\right)$$



$$E_N(\rho(t)) = \log_2\left(\tfrac{3}{4}|1+e^{-t}| + |\tfrac{1}{4} - \tfrac{3}{4}e^{-t}|\right)$$



② $\underline{1}$

$$\exp(i\phi u) = \sum_{k=0}^{\infty} \frac{1}{k!}(i\phi u)^k$$

$$= \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!}\phi^{2k}(u^2)^k + \sum_{k=0}^{\infty} \frac{i^{2k+1}}{(2k+1)!}\phi^{2k+1}u^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{(2k)!}\phi^{2k}\,\mathbb{1} + i\sum_{k=0}^{\infty} \frac{i^{2k}}{(2k+1)!}\phi^{2k+1}\,u$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{(2k)!}\phi^{2k}\,\mathbb{1} + i\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k+1)!}\phi^{2k+1}\,u$$

$$= \cos(\phi)\,\mathbb{1} + i\sin(\phi)\,u$$

② $R_z(\phi) = e^{i\phi Z}$

$$\rho' = R_z(\phi)\rho R_z^\dagger(\phi)$$

$$= (\cos\phi\, I + i\sin\phi\, Z)\rho(\cos\phi\, I - i\sin\phi\, Z)$$

$$= \cos^2\phi\,\rho - i\cos\phi\sin\phi\,\rho Z + i\cos\phi\sin\phi\, Z\rho + \sin^2\phi\, Z\rho Z$$

$$= \frac{\cos^2\phi}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z) - \frac{i\cos\phi\sin\phi}{2}(Z - r_x\sigma_y + r_y\sigma_x + r_z)$$

$$+ \frac{i\cos\phi\sin\phi}{2}(Z + r_x\sigma_y - r_y\sigma_x + r_z) + \frac{\sin^2\phi}{2}(I - r_x\sigma_x - r_y\sigma_y + r_z\sigma_z)$$

$$= \frac{I}{2} + \frac{\cos 2\phi\, r_x \sigma_x - i2\sin\phi\cos\phi\, r_y \sigma_x}{2}$$

$$+ \frac{\cos 2\phi\, r_y \sigma_y + i2\sin\phi\cos\phi\, r_x \sigma_y}{2} + \frac{r_z}{2}\sigma_z$$

$$= \frac{1}{2}\Big(I + (\cos 2\phi\, r_x - i\sin 2\phi\, r_y)\sigma_x$$

$$+ (\cos 2\phi\, r_y + i\sin 2\phi\, r_x)\sigma_y + r_z\sigma_z\Big)$$

z-component of Bloch vector $\vec{r}'$ remains same as $\vec{r}$. $\Rightarrow R_z(\phi)$ rotates Bloch vector around z-axis.

3. A general 2×2 unitary can be written as,

$$U = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \quad \text{where } |a|^2 + |b|^2 = 1$$

Alternatively,

$$U = \begin{pmatrix} \cos\beta\, e^{i\alpha_1} & \sin\beta\, e^{i\alpha_2} \\ -\sin\beta\, e^{-i\alpha_2} & \cos\beta\, e^{-i\alpha_1} \end{pmatrix}$$

Let $\alpha_1 = \alpha + \gamma$
$\alpha_2 = \alpha - \gamma$.

$$\Rightarrow U = \begin{pmatrix} \cos\beta\, e^{i(\alpha+\gamma)} & \sin\beta\, e^{i(\alpha-\gamma)} \\ -\sin\beta\, e^{i(-\alpha+\gamma)} & \cos\beta\, e^{-i(\alpha+\gamma)} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}$$

$$= R_z(\alpha) R_y(\beta) R_z(\gamma) \quad \square$$

3 (N&C)

Any unitary can be written as, (Problem 2).
$U = R_z(\alpha) R_y(\beta) R_z(\gamma)$

Let $A = R_z(\alpha) R_y(\beta/2)$

$B = R_y(-\beta/2) R_z\left(-\frac{\gamma+\alpha}{2}\right)$

$C = R_z\left(\frac{\gamma-\alpha}{2}\right)$

1. $ABC = R_z(\alpha) R_y(\beta/2) R_y(-\beta/2) R_z\left(-\frac{\gamma+\alpha}{2}\right) R_z\left(\frac{\gamma-\alpha}{2}\right)$

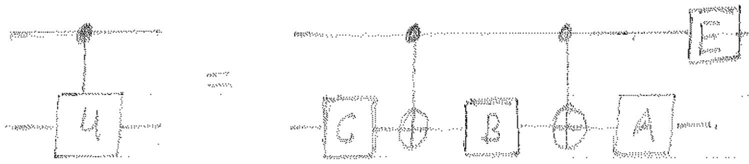
$= R_z(\alpha) R_y(0) R_z\left(\frac{-\gamma-\alpha+\gamma-\alpha}{2}\right)$

$= R_z(\alpha) R_z(-\alpha) = \mathbb{1}$

2. $AXBXC = R_z(\alpha) R_y(\beta/2)\, x\, R_y(-\beta/2) R_z\left(\frac{-\gamma+\alpha}{2}\right) x\, R_z\left(\frac{\gamma-\alpha}{2}\right)$

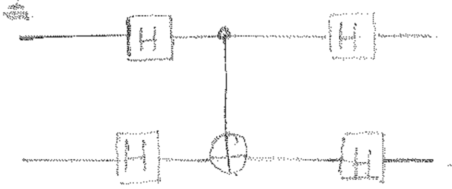
$= R_z(\alpha) R_y\left(\frac{\beta}{2}\right) R_y\left(\frac{\beta}{2}\right) R_z\left(\frac{\gamma+\alpha}{2}\right) R_z\left(\frac{\gamma-\alpha}{2}\right)$

$= R_z(\alpha) R_y(\beta) R_z(\gamma) = U$



$E = \begin{pmatrix} 1 & \\ & e^{i\delta} \end{pmatrix}$

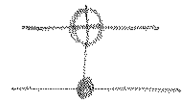
4 1.



$= (H \otimes H)(CNOT)(H \otimes H)$

$= (H \otimes H) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} (H \otimes H)$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 10| + |11\rangle\langle 01|$$

$=$ 

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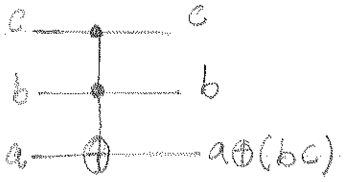

$$= \left(\begin{array}{cccccc|cc} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ \hline &&&&&& 0 & 1 \\ &&&&&& 1 & 0 \end{array}\right)$$

3 $V^2 = \frac{(1-i)^2}{4}(I + iX)^2 = X$

Input			output		
a	b	c	a′	b′	c′
0	0	0	0	0	0
1	0	0	1	0	0
0	0	1	$V^\dagger V\lvert 0\rangle = \lvert 0\rangle$	0	1
1	0	1	$V^\dagger V\lvert 1\rangle = \lvert 1\rangle$	0	1
0	1	0	$VV^\dagger\lvert 0\rangle = \lvert 0\rangle$	1	0
1	1	0	$VV^\dagger\lvert 1\rangle = \lvert 1\rangle$	1	0
0	1	1	$V^2\lvert 0\rangle = \lvert 1\rangle$	1	1
1	1	1	$V^2\lvert 1\rangle = \lvert 0\rangle$	1	1

4

$$C\text{-}C\text{-}V^2 = \left(\begin{array}{cccccc|c} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ \hline & & & & & & V^2 \end{array}\right)$$

$$= \overset{c\,b\,a}{|000\rangle}\langle 000| + |001\rangle\langle 001| + |010\rangle\langle 010| + |011\rangle\langle 011| +$$
$$|100\rangle\langle 100| + |101\rangle\langle 101| + |11\rangle\langle 11| \left(V^2_{00}|0\rangle\langle 0| + V^2_{01}|0\rangle\langle 1| + V^2_{10}|1\rangle\langle 0| + V^2_{11}|1\rangle\langle 1| \right)$$

5 $\boxed{\text{Exercise Sheet 7. Prob. 5}}$