Exercise Sheet 6

Quantum Information

To be returned no later than June 5, 2015

(20 points) **Problem 1: (Log) Negativity.**
The *negativity* of a subsystem A of a systems described by a density matrix ρ_{AB} is defined as

$$\mathcal{N}(\rho) = \frac{1}{2}(\|\rho^{T_A}\|_1 - 1),$$

where trace-norm of operator X, $\|X\|_1 = \sum_j |\lambda_j|$ is the sum of the absolute value of its eigenvalues.
1) Show that $\mathcal{N}$ is a convex function, i.e. $\mathcal{N}\Big(\sum_j p_j \rho_j\Big) \leq \sum_j p_j \mathcal{N}(\rho_j)$.
The *logarithmic negativity* is defined as

$$E_N(\rho) = \log_2 \|\rho^{T_A}\|_1.$$

2) Show that E_N is additive, i. e. $E_N(\rho_1 \otimes \rho_2) = E_N(\rho_1) + E_N(\rho_2)$.
Hint: First show that $\|\rho_1 \otimes \rho_2\|_1 = \|\rho_1\|_1 \|\rho_2\|_1$ using the expression of the trace norm via eigenvalues. Then show that the partial transposition commutes with taking tensor products.

Recall Problem 2 from previous Exercise Sheet 5. Suppose that state $\rho(0) = |\Phi^+\rangle\langle\Phi^+|$ evolves as

$$\rho(t) = p_+ |00\rangle\langle 00| + p_- |01\rangle\langle 01| + p_- |10\rangle\langle 10| + p_+ |11\rangle\langle 11| + e^{-t/T_2}/2\,|00\rangle\langle 11| + e^{-t/T_2}/2\,|11\rangle\langle 00|,$$

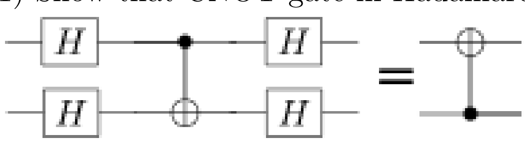
with $p_\pm = \frac{1}{4}(1 \pm e^{-t/T_1})$. Let us take $T_1 = T_2 = 1$.
3) Calculate the negativity of this state $\mathcal{N}(\rho(t))$ and sketch its dependence on time.
4) Calculate the logarithmic negativity $E_N(\rho(t))$ and sketch its dependence on time.

(20 points) **Problem 2: Unitaries.**
1) Show that for any U such that $U^2 = I$ the following holds $\exp\{i\phi U\} = \cos\phi\, I - i \sin\phi\, U$.
2) Verify that $R_z(\phi)$ is indeed rotates a vector $\hat{r} = (r_x, r_y, r_z)$ around z-axis by angle ϕ, i.e. let ρ has a Bloch vector $\hat{r}$, find Bloch vector of a rotated state $\rho' = R_z(\phi)\rho R_z(\phi)^\dagger$.
3) Show that up to a global phase any unitary one-qubit transformation U can be implemented with three rotations about x and z-axes, i.e. find angles α, β, γ and α', β', γ' such that $U = R_x(\alpha)R_z(\beta)R_x(\gamma)$ and $U = R_z(\alpha')R_x(\beta')R_z(\gamma')$. Hint: Up to a global phase factor any unitary transformation on a single qubit is a rotation $U = R_{\hat{n}}(\phi)$ by an angle ϕ about axis $\hat{n} = (n_x, n_y, n_z)$.

There is nothing specific about the choice of x and z axes, one may choose y and z instead, i.e. for some angles α, β, γ the following holds $U = R_z(\alpha) R_y(\beta) R_z(\gamma)$.

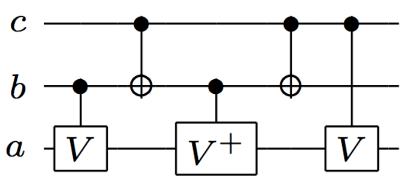
(20 points) **Problem 3: Controlled-U gate.**
In this exercise we will show that for any unitary matrix U controlled-U gate can be realized using only one-qubit and CNOT gates.
1) Use previous exercise to show that for a special unitary matrix $U \in SU(2)$ (i.e. $\det(U) = 1$) there exist matrices $A, B, C \in SU(2)$ such that $ABC = I$ and $AXBXC = U$, where X is one of the Pauli matrices.
2) Find a realization of controlled-U gate (for any unitary U) that uses only matrices A, B, C, CNOT gates and an additional one-qubit gate E that is used to adjust the global phase. Hint: You need to use at most 6 gates.

(20 points) **Problem 4: Gates.**
1) Show that CNOT gate in Hadamard basis flips the roles of target and control qubits:



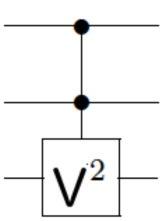
2) Consider Toffoli gate with two control qubits b and c and a target qubit a. The gate performs the following operation: it maps $\{a, b, c\}$ onto $\{a \,\mathrm{XOR}\, (b \,\mathrm{AND}\, c), b, c\}$, where XOR gate is an addition modulo 2 and AND gate is a "multiplication" (the only time a non-zero result occurs is when both inputs are 1). Find matrix representation of Toffoli gate.

3) Show that the Toffoli gate can be realized using CNOT and C-V gates is the following way, setting $V = \frac{1-i}{2}(I + iX)$



Hint: While you can write the corresponding 8x8 matrices, it is easier to consider the action of the circuit for each classical setting of b and c separately, without writing out its matrix representation.

4) In the circuit above let V be any unitary gate, find matrix representation of this transformation and describe what it does. This transformation is called control-control-V^2 gate and is represented in the following way.



(20 points) **Problem 5: Multi-control Toffoli gate.**

A multiple-control Toffoli (MCT) gate with target line t and control lines $(c_1, ..., c_n)$ maps $\{c_1, ..., c_2, t\}$ to $\{c_1, ..., c_n, t \, \text{XOR} \, (c_1 \, \text{AND} \cdots \text{AND} \, c_n)\}$. The MCT gate with three control lines can be realized with the regular (2 control lines) Toffoli gates given one additional ancillary line a. Find this representation. In other words, you are given three control lines c_1, c_2, c_3, one ancillary line a and a target line t. Using only regular Toffoli gates construct a circuit that leaves control and ancillary lines unchanged and maps target line t to $t \, \text{XOR} \, (c_1 \, \text{AND} \cdots \text{AND} \, c_n)$ (independent of ancillas).