Exercise Sheet 10

Quantum Information

To be handed in by July 2nd, 2015

Problem 1: Stabilizers for the 7-qubit code. (25 points)

1. Verify that the following generators stabilize the codewords for the 7-qubit Steane code

$$
\begin{aligned}
g_1 &= IIIXXXX \\
g_2 &= IXXIIXX \\
g_3 &= XIXIXIX \\
g_4 &= IIIZZZZ \\
g_5 &= IZZIIZZ \\
g_6 &= ZIZIZIZ
\end{aligned}
$$

2. What are the logical operators of the code?
3. Recall that the distance of a stabilizer code is defined to be the minimum weight of any element of $\mathcal{N} - \mathcal{S}$. Using this definition, find the distance of 7-qubit Steane code.
4. The stabilizers above closely resemble the parity check matrix of the $[7, 4, 3]$ Hamming code. Show that for any classical code C (with parity check matrix H) which satisfies $C^\perp \subset C$, the CSS code constructed using $C_1 \equiv C$ and $C_2 \equiv C_1^\perp$ has stabilizers which are constructed in the same way from H.

Problem 2: Syndrome measurement and correction for stabilizer codes. (20 points)

Show that the syndrome measurement and the error correction for stabilizer codes can be carried out using only CNOT, H, measurements in the Z basis, and ancillas (and possibly classical side processing). Give the circuit for the syndrome measurement for the 5-qubit code, and explain how the error correction is carried out using classical side processing.

Problem 3: Clifford circuits. (30 points)

Clifford circuits are circuits which are built from $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$, H, and CNOT. In this problem, we will show that a quantum computer which consists only of Clifford gates and Z measurements, and starting from the $|0 \cdots 0\rangle$ state, can be simulated efficiently classically. The core idea is that at each state of the computation, the state of the system is a stabilizer state which can be kept described efficiently through its stabilizers (which can be updated efficiently in any step of the computation).

1. Show that the gate set above allows to obtain all Pauli matrices.
2. Show that Clifford circuits C map products of Paulis $P_1 \otimes \cdots \otimes P_n$ ($P_i = I, X, Y, Z$) to products of Paulis, $C(P_1 \otimes \cdots \otimes P_n)C^\dagger = P_1' \otimes \cdots \otimes P_n'$. Explain why this maps independent stabilizers to independent stabilizers.
3. In each step, we want to describe a unique state, i.e., for n qubits we have n independent stabilizers. Show that implies that for any Pauli product O which commutes with the stabilizers, O or $-O$ is in the stabilizer.
4. Write a (minimal) set of stabilizers for the state $|0\cdots 0\rangle$.
5. Consider a quantum computation consisting of a sequence of Clifford gates $C_1, \ldots, C_\ell$, starting in the state $|\psi_0\rangle = |0\cdots 0\rangle$. Show that in each step of the computation, the state $|\psi_s\rangle = C_s |\psi_{s-1}\rangle$ of the quantum computer can be described by a set of stabilizers, and that the stabilizers for step s can be efficiently computed from those for step $s-1$ (given a C_s is a one- or two-qubit gate).
6. Finally, let us consider Z measurements. W.l.o.g., we will assume that we measure the first qubit.
   a) Show that after the measurement of the first qubit, we are in an eigenstate of $ZI\cdots I$.
   b) Show that if $\pm ZI\cdots I$ is contained in the stabilizer, there exists a minimal basis of stabilizers which contains $\pm ZI\cdots I$, while all other stabilizers are of the form $\pm I*\cdots*$ (where $*$ can be arbitrary Paulis.) Show that this implies that the state is a product state of the first (measured) qubit and the remaining ones, $|i\rangle|\psi'\rangle$, i.e., we can discard the first qubit. What are the new stabilizer for $|\psi'\rangle$?
   c) Consider first the case where $\pm ZI\cdots I$ is contained in the stabilizer *before* the measurement. What is the measurement outcome of a Z measurement on the first qubit? What is the new stabilizer?
   d) Second, consider the case where $\pm ZI\cdots I$ is not contained in the stabilizer.
   - Show that if $\pm ZI\cdots I$ is not contained in the stabilizer, it must anti-commute with at least one stabilizer, since we have n independent stabilizers.
   - Next, show that we can find a minimal basis of stabilizers which only contains a single stabilizer $\hat{S}$ which anti-commutes with $ZI\cdots I$ (i.e., which has a X or Y on the first qubit); in the following, we will work in that basis.
   - Use the existence of this $\hat{S}$ to show that $\langle\psi| ZI\ldots I|\psi\rangle = 0$, i.e., the measurement outcome is completely random.
   - Given a the measurement outcome 0 or 1, we are in an eigenstate of $S_{\text{new}} = \pm ZI\cdots I$, respectively, i.e., S_{new} is a stabilizer for the post-measurement state. Furthermore, all other stabilizers except $\hat{S}$ are still stabilizers, since they commute with S_{new}. Explain how this allows us to obtain n independent stabilizers for the post-measurement state.
7. Put these steps together to explain how quantum computation with Clifford gates can be classically simulated.

It is worth noting that all we need to do in the classical simulation is arithmetics modulo 2,

which is even much weaker than general polynomial-time classical computation; in fact, it is in a complexity class called $\oplus L$ ("parity L"). Thus, quantum computation with Clifford gates is even weaker than classical computation.
Note that adding either initial states or measurements in the eigenbasis of $T = \left(\begin{smallmatrix} 1 & \\ & \exp[i\pi/4] \end{smallmatrix}\right)$ enables us to implement T gates and thus to obtain universal quantum computation. If you want, you can try to figure out how this works. Ideas from Problem 4 might be helpful for that.

Problem 4: Gate teleportation. (25 points)

In gate teleportation, a unitary gate U is applied to a state $|\psi\rangle$ by first preparing the state $|\chi\rangle = (I \otimes U)|\Omega\rangle$, where $|\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (the Choi-Jamiolkowski state of $\mathcal{E} : \rho \mapsto U\rho U^\dagger$), and then "teleporting $|\psi\rangle$ through $|\chi\rangle$", i.e., projecting $|\psi\rangle$ and the left qubit of $|\chi\rangle$ onto the Bell basis. (This can be advantageous if U is difficult to apply or e.g. can only be implemented probabilistically, since $|\chi\rangle$ can be prepared beforehand.)

1. Check that if the measurement outcome is $|\Omega\rangle$, the output state (i.e., the right qubit of $|\chi\rangle$) is $U|\psi\rangle$. (*Note:* You might recall that this is just the Choi-Jamiolkowski isomorphism backwards!)
2. What is the output for each of the other three possible measurement outcomes?
3. Now consider the case where U is a Clifford gate [i.e., a Pauli matrix, H, or $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$]. Show that for each measurement outcome, we can transform the output state to $U|\psi\rangle$ (up to an irrelevant global phase) by afterwards applying a Pauli matrix depending on the measurement outcome.
4. Explain why the steps above (i.e., the Bell measurement and the Clifford gate) can be implemented on qubits encoded with a stabilizer code without decoding them. (You can use the result of the lecture that Clifford gates are exactly those which map Paulis to Paulis.)
5. Now consider $U = T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \exp[i\pi/4] \end{smallmatrix}\right)$. Show that we can transform the output state to $U|\psi\rangle$ by applying a Clifford gate depending on the measurement outcome.
6. Finally, consider gate teleportation for a CNOT gate. Construct the state $|\chi\rangle$ (which is now a state of 2+2 qubits) corresponding to the CNOT gate, and show that $|\chi\rangle$ can be used to implement a deterministic CNOT gate by applying Pauli gates which depend on the measurement outcome.