Exercise Sheet 11

Quantum Information

To be handed in by July 9nd, 2015

Problem 1: Photonic quantum computation. (40 points)

Single photons can be used to implement qubits. One possible encoding is a dual-rail encoding where a single photon can be in one of two different paths, described by creation operators $a^\dagger$ and $b^\dagger$; i.e., the qubit is encoded as $|0\rangle = a^\dagger |\Omega\rangle$ and $|1\rangle = b^\dagger |\Omega\rangle$, where $|\Omega\rangle$ is the vacuum.

1. We can realize single-qubit gates if we couple the two modes on a beam splitter. The action of the beam splitter (mapping the input modes a_{in} and b_{in} and output modes a_{out} and b_{out}) is given by

$$\begin{aligned} a_{\text{out}} &= \cos(\theta)\, a_{\text{in}} - \sin(\theta)\, b_{\text{in}} \\ b_{\text{out}} &= \sin(\theta)\, a_{\text{in}} + \cos(\theta)\, b_{\text{in}} \ , \end{aligned} \tag{1}$$

   where θ is a parameter described the degree of mixing of the beam splitter. Determine the class of single-qubit gates which can be realized by such a beam splitter. (*Note:* Beam splitters are linear optical elements, i.e., their action on any power a^k etc. is given by its action on b.)

2. Another way to implement a gate is by inserting a phase plate in the light path of a, which acts (linearly) as $a_{\text{out}} = e^{i\phi} a_{\text{in}}$. Which gates can we realize this way?

3. Now let us consider two qubits, given by modes a_1, b_1 and a_2, b_2, where each pair of modes contains exactly one photon. Consider that we insert a beam splitter with $\theta = \pi/4$ between modes b_1 and b_2.
   a) First, consider the case where only one qubit is in state $|1\rangle$ (i.e., with one photon incident on the beam splitter), and show that this will lead to states which are outside of the qubit subspace, i.e., with more (or less) than one photon in one pair of modes a_k, b_k).
   b) Now consider that we insert a second beam splitter with angle θ' after the first one (acting on the same modes). Show that there is a choice of θ' where the qubit subspace is restored for the case of one incident photon.
   c) Now consider the case of two incident photons (i.e., the input states $|1\rangle_1 |1\rangle_2$). Derive the state after the beam splitter for $\theta = \pi/4$. [Note: It might be convenient to invert the relation (1) and apply it to the input state $b_1^\dagger b_2^\dagger |\Omega\rangle$]. You should find that both photons always exit in the same mode, which is known as the Hong-Ou-Mandel effect.
   d) What happens in the case of two incident photons if we insert a second beam splitter in the path? Show that the qubit subspace is again restored for the right choice of θ'.
   e) Show that this protocol did not implement any (non-trivial) two qubit gate. Show that if we had a sufficiently strong *nonlinear* element which gives a phase shift of $e^{i\pi}$ to $(b^\dagger)^2 |\Omega\rangle$ but not to $b^\dagger |\Omega\rangle$, we could use this to implement a controlled-Z gate.

Problem 2: CNOT gates from exchange interaction. (30 points)

In this problem, we will study how to obtain CNOT gates from exchange interactions of the form

$$H = \sum_{\alpha=x,y,z} J_\alpha \, \sigma_\alpha^1 \otimes \sigma_\alpha^2 + B_1 \, (\sigma_z^1 \otimes I) + B_2 \, (I \otimes \sigma_z^2) \ ,$$

where σ_α^i denotes the α's Pauli matrix acting on site i, by applying the interaction for a time t, i.e., $U = e^{-iHt}$.

1. First, recall how the CNOT can be transformed into a Controlled-Z (CZ) gate using single qubit gates; we will focus on the realization of the CZ gate in the rest of the problem.
2. Show that there is a simple choice of the J_α and B_i (where in fact only one J_α is non-zero) which allows to obtain $CZ \propto e^{-iHt}$, and give J_α, B_i, and t.
3. Now consider the isotropic case where $J_x = J_y = J_z \equiv J$, and find again J, B_i, and t such that e^{-iHt} is equivalent to CZ up to local unitaries. Show that these local unitaries can also be realized using H for a different set of parameters, i.e., $CZ \propto e^{-iH_1 t} e^{-iH_2 t}$, with H_1, H_2 the two sets of parameters.
   (*Hint:* First, diagonalize $\sum \sigma_\alpha^1 \otimes \sigma_\alpha^2$ – note that this interaction is SU(2) invariant, i.e., the eigenvectors can be labelled by S and S_z – and then express the rest of the Hamiltonian in this basis, and use this to determine e^{-iHt}.)

Problem 3: Computing with mixed states. (30 points)

In nuclear magnetic resonance (NMR) computing, computations are carried out in an ensemble of molecules at room temperature.

1. Assume the relevant degrees of freedom of the molecule are described by a Hamiltonian H which is rescaled to be of the order of 1. At room temperature, this gives a very small $\beta \approx 10^{-4}$. Show that the thermal state $\rho = e^{-\beta H}/\mathrm{tr}[e^{-\beta H}]$ is well approximated by
$$\rho = \tfrac{1}{2^n}(I - \beta H) \ .$$
2. Now consider the case of one qubit. Show that in this case, this is close to a "pseudo-pure" state $\rho = \frac{1}{2}(I - \beta \left|\psi\right\rangle \left\langle\psi\right|)$. What is the probability distribution of measurement outcomes in the computational basis $\left|0\right\rangle$, $\left|1\right\rangle$? What is the variance? Now assume $\left|\psi\right\rangle$ is in a computational basis state. How many copies K of ρ do we need to measure in order to be able to separate the two outcomes with good confidence (say, one standard deviation)?
3. Now consider a pseudo-pure state of n qubits, $\rho = \frac{1}{2^n}(I - \beta \left|\psi\right\rangle \left\langle\psi\right|)$. Repeat the previous argument for this case – how does the number of copies we need to measure to distinguish the 2^n different outcomes scale with n (assuming again that $\left|\psi\right\rangle$ is in a computational basis state)?