<u>Note:</u> Seq. meas. of $Z \to X \to Z$ will give a <u>random outcome</u> for 2nd Z meas.

$\Rightarrow$ X and Z cannot be measured simultaneously ($\Rightarrow$ uncertainty!)

Measurement on a bipartite state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Alice measures Z, Bob measures Z:

Projectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

$\Rightarrow$ 01 and 10 w. prob $1/2$

Alice measures X, Bob meas. X:

Projectors $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$:

(use $\langle +|0\rangle = \langle +|1\rangle = \langle -|0\rangle = 1/\sqrt{2}$; $\langle -|1\rangle = -1/\sqrt{2}$)

$$|\langle ++|\psi\rangle|^2 = 0$$

$$|\langle +-|\psi\rangle|^2 = \left|-\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\langle -+|\psi\rangle|^2 = \dots = \frac{1}{2}$$

$$|\langle --|\psi\rangle|^2 = \dots = 0$$

$\Rightarrow$ anticorrelation

In fact, anti-corr in <u>all</u> basis (HW)

Alice meas. X, Bob meas. Z:

$$\left.\begin{aligned} |\langle +0|\psi\rangle|^2 &= \left|-\tfrac{1}{2}\right|^2 = \tfrac{1}{4} \\ |\langle +1|\psi\rangle|^2 &= \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{4} \\ |\langle -0|\psi\rangle|^2 &= \tfrac{1}{4} \\ |\langle -1|\psi\rangle|^2 &= \tfrac{1}{4} \end{aligned}\right\}$$ <u>no correlation!</u>

All meas. for Alice and Bob look random, <u>but</u> their outcomes in the <u>same basis</u> are perfectly anti-correlated!

<u>2. Mixed states</u>

Consider bipartite state $|\psi\rangle_{AB}$. We only have access to A.

e.g. $|\psi\rangle_{AB} = c_{00}|0\rangle|0\rangle + c_{01}|0\rangle|1\rangle + c_{10}|1\rangle|0\rangle + c_{11}|1\rangle|1\rangle$.

Can we characterize measurement outcomes on A in a simple way?

A measures $M \longrightarrow$ Joint measurement $M_A \otimes I_B$

$$\langle\psi|M_A \otimes I_B|\psi\rangle = \sum c^*_{i'j'} \langle i'|\langle j'| (M_A \otimes I_B)\, c_{ij} |i\rangle|j\rangle$$

$$= \sum c^*_{i'j'} c_{ij} \langle i'|M_A|i\rangle \underbrace{\langle j'|j\rangle}_{=\delta_{jj'}}$$

$$= \sum' \langle i'|\Pi|i\rangle\, c_{i'j}^{*}\, c_{ij}$$

$$= \mathrm{tr}\left[\, \Pi \left(\sum |i\rangle\langle i'|\, c_{ij}\, c_{i'j}^{*}\right)\right] = \underline{\mathrm{tr}\left[\Pi \rho_A\right]}$$

(*) with $\rho_A = \sum' c_{ij} c_{i'j}^{*} |i\rangle\langle i'|$ the <u>density operator</u> or <u>density matrix</u>

and the <u>trace</u> $\mathrm{tr}[X] = \sum' \langle k|X|k\rangle$.

Note that the trace is <u>cyclic</u>:

$$\underline{\mathrm{tr}[\overset{\leftarrow \text{non-square}}{AB}]} = \sum_k \langle k|AB|k\rangle = \sum_{kl} \langle k|A|l\rangle\langle l|B|k\rangle$$

$$= \sum \langle l|B|k\rangle\langle k|A|l\rangle = \underline{\mathrm{tr}[BA]}.$$

<u>Density operator</u>: Characterization of system w/ only partial knowledge.

What are the properties of ρ_A? Use (*):

$$\rho_A^{+} = \sum \left(c_{ij} c_{i'j}^{*}\right)^{*} |i'\rangle\langle i| = \sum c_{i'j}\, c_{ij}^{*}\, |i'\rangle\langle i| = \rho_A$$

$$\langle\phi|\rho_A|\phi\rangle = \sum c_{ij} c_{i'j}^{*} \underbrace{\langle\phi|i\rangle}_{a_i^{*}} \underbrace{\langle i'|\phi\rangle}_{a_{i'}}$$

$$= \sum_j \left(\sum_i a_i^{*} c_{ij}\right)\left(\sum_{i'} a_{i'} c_{i'j}^{*}\right) = \sum' w_j^{*} w_j \geq 0.$$

$\Rightarrow \rho_A \geq 0$ (i.e., ρ_A has only non-negative eigenvalues: it is <u>positive semidefinite</u>)

$$\mathrm{tr}[\rho_A] = \sum_k \sum_{i i' j} c_{ij} c_{i'j}^* \underbrace{\langle k|i\rangle\langle i'|k\rangle}_{=\delta_{ik}\delta_{i'k}} = \sum c_{kj} c_{kj}^* = 1.$$

<u>Properties of density operators:</u>

- $\rho_A^\dagger = \rho_A$
- $\rho_A \geq 0$
- $\mathrm{tr}(\rho_A) = 1.$

(Note: the ρ_A form a convex set S, i.e.:
$\rho, \sigma \in S \Rightarrow p\rho + (1-p)\sigma \in S$, $0 \leq p \leq 1$)

<u>We will see:</u> This provides an <u>alternative fundamental</u> definition of a state (i.e., <u>all</u> ρ_A of the above form arise if we only have access to part of a system.)

If state of A is <u>pure</u>, i.e., $|\psi\rangle = |\phi_A\rangle \otimes |\chi_B\rangle$

$$\Rightarrow \rho_A = |\phi_A\rangle\langle\phi_A|$$

(can be seen e.g. by writing everything in a basis which contains $|\phi_A\rangle$ and $|\chi_B\rangle$, resp., and use basis independence.)

Note: For a given state $|\psi\rangle_{AB}$, the ρ_A for which

$$\mathrm{tr}[\Pi \rho_A] = \langle\psi| \Pi \otimes I |\psi\rangle$$

is uniquely determined (since its Hilbert-Schmidt scalar product w. all Hermitian Π is determined).

$\Rightarrow$ all numbers in ρ_A are meaningful (unlike the phase of a pure state vector) — "nicer" than for pure states

Partial trace:

Imagine A+B are mixed: ρ_{AB}.

Description of measurement Π on A?

$$\mathrm{tr}\left[(\Pi \otimes I)\rho_{AB}\right] = \sum \underbrace{\langle ij| \Pi \otimes I |i'j'\rangle}_{\delta_{jj'}} \langle i'j'| \rho_{AB} |ij\rangle$$

$$= \sum \langle i| \Pi |i'\rangle \langle i'j| \rho_{AB} |ij\rangle$$

$$= \mathrm{tr}\left[\Pi \cdot \rho_A\right]$$

with $\rho_A = \sum |i'\rangle\langle i'j| \rho_{AB} |ij\rangle\langle i|$

$= \ldots$

$$= \sum \mathbb{1}_A \otimes \langle j|_B \, (\rho_{AB}) \; \mathbb{1}_A \otimes |j\rangle_B$$

$$= \sum \langle j|_B \, \rho_{AB} \, |j\rangle_B$$

$$= \mathrm{tr}_B \, \rho_{AB} \quad : \quad \text{"partial trace"}$$

In components:

$$\left(\mathrm{tr}_B \, \rho_{AB}\right)_{ii'} = \sum_j \left(\rho_{AB}\right)_{(ij)(i'j)}$$

Interpretation of density matrix:

Consider $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$

$$\Rightarrow \quad \rho_A = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} = |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$$

$$\mathrm{tr}\left[\Pi \rho_A\right] = |\alpha|^2 \langle 0|\Pi|0\rangle + |\beta|^2 \langle 1|\Pi|1\rangle$$

$\Rightarrow$ can be interpreted as having $|0\rangle$ w/ $p_0 = |\alpha|^2$ and $|1\rangle$ w/ $p_1 = |\beta|^2$: "ensemble interpretation"

But: State of A+B pure : consistent interpretation?

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B can prepare ensemble by proj. meas. in Z basis:

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle \quad \xrightarrow[\text{on } B]{Z \text{ meas.}} \quad \begin{cases} p_0 = |\alpha|^2: & |\psi_0\rangle_A = |0\rangle_A \\ p_1 = |\beta|^2: & |\psi_1\rangle_A = |1\rangle_A \end{cases}$$

$\Rightarrow$ ensemble $\mathcal{J}_A = \{(p_0, |0\rangle), (p_1, |1\rangle)\}$ for Alice.

Bob knows which state Alice holds!

$\rightarrow$ Bob's description of Alice's state is either $|0\rangle$ or $|1\rangle$, not ρ_A!

$\rightarrow$ Description of q. state depends on knowledge!

But: Bob could do a different measurement, e.g. in the $|\pm\rangle$ basis!

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle \quad \xrightarrow[\text{on } B]{X \text{ meas.}} \quad \begin{cases} p_+ = \frac{|\alpha|^2 + |\beta|^2}{2} = \frac{1}{2}: & |\psi_+\rangle_A = \frac{\alpha|0\rangle + \beta|1\rangle}{|\alpha|^2 + |\beta|^2} \\ p_- = \frac{|\alpha|^2 + |\beta|^2}{2} = \frac{1}{2}: & |\psi_-\rangle_A = \frac{\alpha|0\rangle - \beta|1\rangle}{|\alpha|^2 + |\beta|^2} \end{cases} \text{ non-orthogonal!}$$

$$\rho_A = p_+ |\psi_+\rangle\langle\psi_+| + p_- |\psi_-\rangle\langle\psi_-|$$

$\Rightarrow$ Different ensemble interpretation of same state.

$\Rightarrow$ ambiguity of ensemble interpretation!

In fact, there are infinitely many ensembles for the same density operator!

Even the number of terms can vary:

E.g.: $|\psi\rangle = \frac{\alpha|00\rangle + \beta|11\rangle}{\sqrt{2}} + \frac{\alpha|02\rangle + \beta|13\rangle}{\sqrt{3}}$

Measure in basis $|0\rangle, |1\rangle, \frac{|2\rangle \pm |3\rangle}{\sqrt{2}} =: |\pm\rangle$

$\rightarrow$ combination of the two previous schemes!

$$\Rightarrow \quad \rho_A = \underset{|\alpha|^2/2}{\underset{\shortparallel}{p_0}} |0\rangle\langle 0| + \underset{|\beta|^2/2}{\underset{\shortparallel}{p_1}} |1\rangle\langle 1| + \underset{1/4}{\underset{\shortparallel}{p_+}} |\psi_+\rangle\langle\psi_+| + \underset{1/4}{\underset{\shortparallel}{p_-}} |\psi_-\rangle\langle\psi_-|$$

How are different ensembles related?

<u>Theorem</u>: Let $\rho = \sum p_i |\psi_i\rangle\langle\psi_i| = \sum q_j |\phi_j\rangle\langle\phi_j|$.

Then, there is a unitary $U = (u_{ij})$ s.th.

$$\sqrt{p_i}|\psi_i\rangle = \sum u_{ij} \sqrt{q_j}\, |\phi_j\rangle,$$

and vice versa. (If the number of i's and j's is different, pad the smaller one with zero vectors!)