$\Rightarrow$ can be extended to unitary U

$$
\begin{array}{c c}
 & |0\rangle_B \\
\begin{array}{c} \langle 0|_B \\ \langle 1|_B \\ \vdots \\ \langle d|_B \end{array} & U = \begin{pmatrix} M_0 & \cdots \\ M_1 & \cdots \\ \vdots & \\ M_{d-1} & \cdots \end{pmatrix}
\end{array}
$$

i.e.: $\quad \langle n|_B \, U \, |0\rangle_B = M_n .$

$\Rightarrow$ measurement $\{M_n\}$ can be realized by unitary U & projective measurement!

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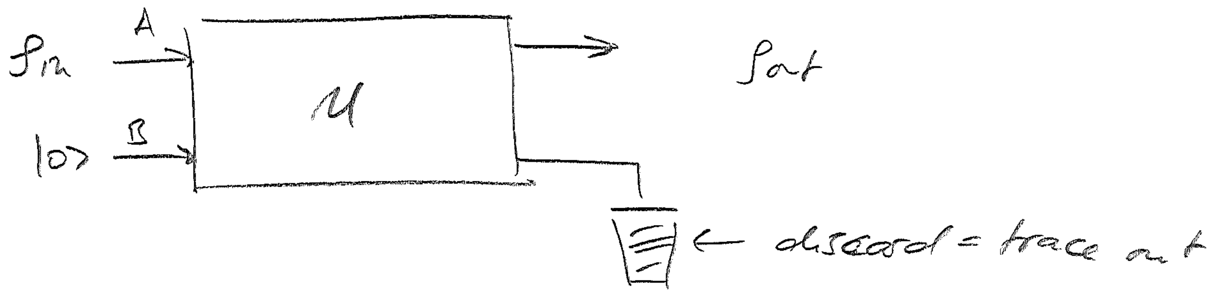
5. General evolution – superoperators

What evolutions can we realize by evolving a larger system with a unitary?



$$\rho \mapsto \mathcal{E}(\rho) = \mathrm{tr}_B\left[\mathcal{U}(\rho \otimes |0\rangle\langle 0|)\,\mathcal{U}^\dagger\right]$$

$$= \sum' \langle u|_B\, \mathcal{U}\, |0\rangle_B\ \rho\ \langle 0|_B\, \mathcal{U}\, |u\rangle_B$$

$$= \sum M_u \rho M_u^\dagger \quad ; \quad \text{with} \quad M_u = \langle u|_B\, \mathcal{U}\, |0\rangle_B .$$

(Note: Trace in any basis $|\tilde{m}\rangle = \sum' v_{mu}|u\rangle$, v unitary:

M_u and $\tilde{M}_m = \sum v^\dagger_{mu}\, M_u$ describe <u>same evolution</u>.)

Properties of M_u?

As before: $\sum M_u^\dagger M_u = \sum \langle 0|_B\, \mathcal{U}\, |u\rangle_B \langle u|_B\, \mathcal{U}^\dagger\, |0\rangle_B = \mathbb{1}_A$.

> We call a <u>physical map</u> between density matrices a <u>superoperator</u>, and the form
>
> $$\mathcal{E}: \rho \mapsto \sum M_u \rho M_u^\dagger ; \quad \sum M_u^\dagger M_u = \mathbb{1}$$
>
> its <u>Kraus representation</u>.

<u>Note:</u> Any map of the form $\rho \mapsto \sum M_u \rho M_u^\dagger$ can be implemented via unitary + tracing out (cf. POVMs). In fact, $\mathcal{E}$ can be considered as a POVM meas. w/out knowing the result ("measurement by environment")

What is the most general physical evolution $\mathcal{E}$?

Properties:

- hermiticity-preserving: $\rho = \rho^\dagger \Rightarrow \mathcal{E}(\rho) = \mathcal{E}(\rho^\dagger)$
- positive: $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) \geq 0.$
- trace-preserving: $\mathrm{tr}(\rho) = 1 \Rightarrow \mathrm{tr}(\mathcal{E}(\rho)) = 1$
- linearity: $\mathcal{E}(\rho + \lambda\sigma) = \mathcal{E}(\rho) + \lambda\,\mathcal{E}(\sigma).$

Do we need linearity?

$\rightarrow$ Yes, otherwise ensemble interpretation becomes inconsistent! ($\rightarrow$ Homework)

Is this enough for a physical map?

No. Want that $\mathcal{E}$ acts as a physical map even on part of a larger system:

"complete positivity"

$$\rho_{AB} \geq 0 \Rightarrow (\underline{\mathcal{E}_A \otimes \mathbb{1}_B})(\rho_{AB}) \geq 0$$

Note: $\mathcal{E}_A \otimes \mathbb{1}_B$ is defined using linearity on a basis:

$$(\mathcal{E}_A \otimes \mathbb{1}_B)(M \otimes N) = \mathcal{E}_A(M) \otimes N.$$

We call $\mathcal{E}$ a completely positive trace preserving (CPTP) map (or "quantum channel").

Are there maps which are positive but not CP?

Yes: E.g. "transpose channel":

$$\mathcal{E} : \rho \longmapsto \rho^T$$

$$(\mathcal{E} \otimes \mathbb{1})(\rho_{AB}) = \rho^{T_A} \quad ; \text{"partial transposition"}$$

E.g.: $|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$

$$(|\Omega\rangle\langle\Omega|)^{T_B} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

T_B

$$\hat{=} \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \not\geq 0.$$

Note: Positive but not completely positive maps are very important as "entanglement witnesses": $(\mathcal{E}\otimes\mathbb{1})(\rho) \geq 0$ for all unentangled states, but $(\mathcal{E}\otimes\mathbb{1})(\sigma) \not\geq 0$ can "witness" certain entangled states σ.

Are all CPTP maps of Kraus form?

The Choi-Jamiołkowski isomorphism:

Let $\mathcal{E}$ be CPTP.

Define $\sigma_{AB} = (\mathcal{E} \otimes \mathbb{1})(|\Omega\rangle\langle\Omega|)$; $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle|i\rangle$.

Then, $\sigma_{AB} \geq 0$, and

$$\mathrm{tr}_A(\sigma_{AB}) = \frac{1}{d}\sum \underbrace{\mathrm{tr}_A(\mathcal{E}_A(|i\rangle\langle j|)}_{=\delta_{ij}} \otimes |i\rangle\langle j|_B) = \frac{1}{d}\mathbb{1}$$

the "Choi-Jamiołkowski state" of $\mathcal{E}$.

We have $d\,\mathrm{tr}_B[\sigma_{AB}\,\rho_B^T] = \sum_{ij} \mathrm{tr}_B[\mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j|\,\rho_B^T]$

$$= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \cdot \underbrace{\langle j|\rho_B^T|i\rangle}_{=\rho_{ij}} = \mathcal{E}(\rho_B)$$

and $\mathrm{tr}_A\left(d\,\mathrm{tr}_B[\sigma_{AB}\,\rho_B^T]\right) = d\,\mathrm{tr}_B\left[\mathrm{tr}_A(\sigma_{AB}\,\rho_B^T)\right]$

$$= d\,\mathrm{tr}_B\left(\underbrace{(\mathrm{tr}_A\sigma_{AB})}_{\overset{!}{=}\frac{1}{d}\mathbb{1}}\rho_B^T\right)$$

$\Rightarrow$ trace-preserving iff $\mathrm{tr}_A\,\sigma_{AB} = \frac{1}{d}\mathbb{1}$.

Isomorphism ("Choi-Jamiolkowski isomorphism") between superoperators $\mathcal{E}: \mathcal{B}(\mathbb{C}^d) \longrightarrow \mathcal{B}(\mathbb{C}^d)$

and states $\sigma_{AB} \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$; $\sigma_{AB} \geq 0$.

Moreover: $\mathcal{E}$ T.P. iff $\mathrm{tr}_A \sigma_{AB} = \frac{1}{d}\mathbb{1}$.

Back to: Is any superoperator of Kraus form?

Let σ_{AB} be its C-J state.

$$\Rightarrow \quad \sigma = \sum |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k| \quad \leftarrow \text{unnormalized.}$$

Write $|\tilde{\psi}_k\rangle = \sum_{ij} m_k^{ij} |j\rangle|i\rangle \overset{\swarrow \text{ONB}}{=} \frac{1}{\sqrt{d}}\sum_i (M_k \otimes \mathbb{1})|i\rangle|i\rangle = (M_k \otimes \mathbb{1})|\Omega\rangle$

$$\Rightarrow \quad \sigma = \sum M_k |\Omega\rangle\langle\Omega| M_k^\dagger \text{; and}$$

$$\mathcal{E}(\rho) = d\, \mathrm{tr}_B\Big[\sum_k (M_k \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(M_k^\dagger \otimes \mathbb{1})(\mathbb{1}\otimes\rho^T)\Big] = d\sum_k M_k \underbrace{\mathrm{tr}_B\big[|\Omega\rangle\langle\Omega|\rho^T\big]}_{\circledast} M_k^\dagger,$$

$$\boxed{\circledast = \frac{1}{d}\sum_{ij} |i\rangle\langle j|\, \underbrace{\mathrm{tr}_B\big[|i\rangle\langle j|\rho^T\big]}_{=\rho_{ij}} = \frac{1}{d}\rho}$$

$$= \sum_k M_k \rho M_k^\dagger.$$

Furthermore:

$$\mathbb{1} = \mathrm{tr}_A \sigma = \mathrm{tr}_A \sum_k (\Pi_k \otimes \mathbb{1}) |\Omega\rangle\langle\Omega| (\Pi_k^\dagger \otimes \mathbb{1})$$

$$= \sum_k \mathrm{tr}_A (\Pi_k^\dagger \Pi_k \cdot |\Omega\rangle\langle\Omega|)$$

$$= \sum_k \mathrm{tr} (\Pi_k^\dagger \Pi_k |i\rangle\langle j|) \otimes |i\rangle\langle j|$$

$$\Longrightarrow \langle j| \sum_k \Pi_k^\dagger \Pi_k |i\rangle = \delta_{ij}$$

$$\longrightarrow \sum_k \Pi_k^\dagger \Pi_k = \mathbb{1}$$

$\Longrightarrow$ <u>All superoperators</u> are of <u>Kraus form</u>.

$\sum_k \Pi_k^\dagger \Pi_k$ corresponds to trace-preserving.

<u>Note:</u> Non-t.p. maps can be implemented by <u>postselecting</u> on certain <u>measurement outcomes</u>.

6. Axioms ("mixed" version)

- States are linear operators $\rho \in \mathbb{B}(\mathcal{H})$ with

$$\rho = \rho^{+}$$
$$\rho \geq 0$$
$$\mathrm{tr}\,\rho = 0$$

- Evolution is described by trace-preserving completely positive maps

$$\mathcal{E} : \rho \longmapsto \mathcal{E}(\rho) = \sum \Pi_n \rho \Pi_n^{+}$$

$$\text{with } \sum \Pi_n^{+} \Pi_n = \mathbb{1}.$$

- Measurements act as

$$\rho \longrightarrow \rho_n = \frac{\Pi_n \rho \Pi_n^{+}}{\mathrm{tr}(\Pi_n \rho \Pi_n^{+})}$$

with probability $p_n = \mathrm{tr}(\Pi_n^{+} \Pi_n \rho)$,

and $\sum \Pi_n^{+} \Pi_n = \mathbb{1}$.