# III. Entanglement



## 1. Introduction

Consider bipartite pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

If $\underset{\phantom{x}}{|\psi_P\rangle} = \underset{\mathcal{H}_A}{\underset{\cap}{|\phi^A\rangle}} \otimes \underset{\mathcal{H}_A}{\underset{\cap}{|\phi^B\rangle}}$ :

A & B can describe all their measurements etc.

on $|\psi\rangle$ independently $\rightarrow$ no correlations.

We call such a state a *product state.*

Product states have Schmidt coefficients $(1, 0, \ldots)$,

and $\rho_A = \mathrm{tr}_B |\psi_P\rangle\langle\psi_P| = |\phi^A\rangle\langle\phi^A|$,

$\rho_B = \mathrm{tr}_A |\psi_P\rangle\langle\psi_P| = |\phi^B\rangle\langle\phi^B|$

are pure states (i.e., rank-1 projectors).

$\Longleftrightarrow \mathrm{tr}\, \rho_A^{\,2} = \mathrm{tr}\, \rho_B^{\,2} = 1$.

(Note: For general $\rho = \sum p_i |\chi_i\rangle\langle\chi_i|$, $\overset{\text{eigvals}}{\sum p_i} = 1$, we have

$\mathrm{tr}\, \rho^2 = \sum p_i^2 \leq 1$ ) $\boxed{\text{"purity"}}$

We call (pure) states which are not product states **entangled**.

Consider e.g. $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

- Measurement outcomes of A & B are anti-correlated
  $\rightarrow$ no independent description possible.
- $\rho_A = \rho_B = \frac{1}{2}\mathbb{1}$ : all meas. outcomes equally likely.

  $\rightarrow \mathrm{tr}\,\rho_A^2 = \mathrm{tr}\,\rho_B^2 < 1$ for all entangled states!

  $\rightarrow$ ent. states have more than one Schmidt-coeff. $\neq 0$.

Encoding of information:

dim $(\mathbb{C}^2 \otimes \mathbb{C}^2) = 2^2 \Rightarrow 2$ bits!

Product states:

$|\psi_{ij}\rangle = |i\rangle|j\rangle$ : orthonormal set.

$\rightarrow$ encoding in product states.

$\rightarrow$ A & B can read out information individually

Entangled states:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

ONB

"Bell states",
"Bell basis"

→ encodes 2 bits of info

→ How much info can we retrieve w/ indiv. meas.?

A's meas. fully random → no info
→ total info at most 1 bit!

A & B meas both in Z basis:
→ recover if equal ($\phi$'s) or different ($\psi$'s)

Similar w/ X-basis: + or −, etc.

→ 2 bits of info, but only 1 bit can be recovered locally ⟹ info hidden in (non-classical) correlations

→ data hiding schemes!

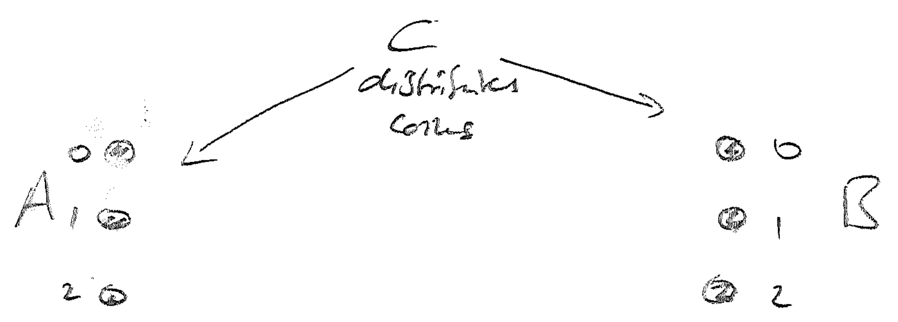
Goals of study of entanglement:

- How non-classical are entangled states?
- What can we do with entangled states? ("resources")
- How can we <u>quantify</u> the amount of entanglement?
- How can we manipulate entanglement?
- What about entanglement in <u>mixed states</u>?

## 2. Bell inequalities

How non-local are entangled states?

Consider the following game of A+B with coins:



- A+B get each a set of 3 coins (0,1,2) prepared in same way by C.
- A and B can only each look at one coin; they get heads $= +1$ or tails $= -1$. Let us denote the result by $a_i = \pm 1$ and $b_j = \pm 1$ $(i,j = 0,1,2)$.
- If A&B look at the same coin, they always get the same result, $a_i = b_i$.
- Can A infer the value of two coins?

  Idea: A looks at $i$; Bob at $j = i' \neq i$.

  Since $a_{i'} = b_{i'}$, they now know $a_i$ and $a_{i'}$.

- What can we say about the probability

$$p(a_i = a_{i'}) \quad ?$$

$$p(a_0 = a_1) + p(a_1 = a_2) + p(a_2 = a_0) \geq 1,$$

since in each instance, at least two coins must be equal.

$$\Rightarrow \quad p(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) \geq 1 \quad !$$

What happens in a quantum version of this experiment? (48)

A & B share an entangled state, and perform proj. measurement along three different axes with outcomes $\pm 1$ $\longrightarrow$ meas operators $a_i$ and $b_j$.

A & B share $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

We have $(\vec{\sigma}^A + \vec{\sigma}^B)|\psi^-\rangle = 0$

$\left[\text{i.e. } (\sigma_i^A + \sigma_i^B)|\psi^-\rangle = 0 \ \forall i, \ \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\right]$

Then, $\langle\psi^-|(\vec{\sigma}^A \cdot \vec{n})(\underbrace{\vec{\sigma}^B}_{= -\vec{\sigma}^A|\psi^-\rangle} \cdot \vec{m})|\psi^-\rangle$

$= -\langle\psi^-|(\vec{\sigma}^A \cdot \vec{n})(\vec{\sigma}^A \cdot \vec{m})|\psi^-\rangle$

$= -\sum_{ij} n_i m_j \underbrace{\mathrm{tr}(\underbrace{\rho_A}_{=\frac{1}{2}\mathbb{1}} \sigma_i^A \sigma_j^A)}_{=\delta_{ij}} = -\sum_i n_i m_i = -\vec{n} \cdot \vec{m}$

$= -\cos\theta$

angle betw. $\vec{n}$ & $\vec{m}$.

Measurement of A/B along $\vec{u} / \vec{m}$:

$\longrightarrow$ projections $E_\pm(\vec{u}) = \frac{1}{2}\left(\mathbb{1} \pm \vec{u}\cdot\vec{\sigma}\right)$

$\underline{P(\pm 1, \pm 1)} = \langle \psi^- | E_\pm(\vec{u})\, E_\pm(\vec{m}) | \psi^- \rangle$

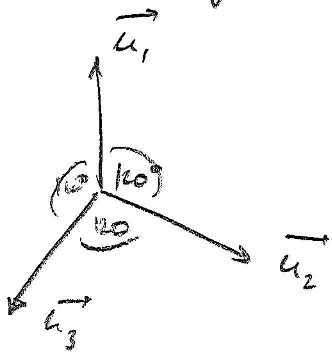
$$= \frac{1}{4} \langle \psi^- | \underbrace{\mathbb{1}}_{\to 1} \pm \underbrace{\vec{u}\cdot\vec{\sigma}_A}_{\to 0} \pm \underbrace{\vec{m}\cdot\vec{\sigma}_B}_{\to 0} + \underbrace{(\vec{u}\cdot\vec{\sigma}_A)(\vec{m}\cdot\vec{\sigma}_B)}_{\to -\cos\Theta} | \psi^- \rangle$$

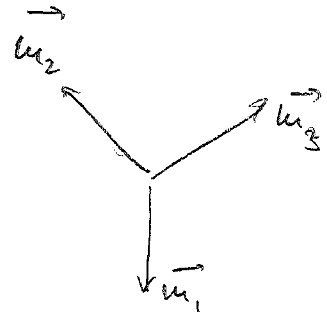
$$= \frac{1}{4}(1 - \cos\Theta)$$

$\underline{P(\pm 1, \mp 1)} = \frac{1}{4}(1 + \cos\Theta)$.

$\Longrightarrow P_{\text{equal}} = \frac{1}{2}(1-\cos\Theta)$; $P_{\text{different}} = \frac{1}{2}(1+\cos\Theta)$

Now let A measure along




in the x-z-plane, and Bob along $\vec{m}_i = -\vec{u}_i$

- A+B measure in same basis:

$$P_{\text{equal}} = \frac{1}{2}(1 - \cos 180°) = 1 \qquad \checkmark$$

- A+B measure in different bases:

$$P_{equal} = \frac{1}{2}\left(1 - \underset{\substack{\| \\ \theta = \pm 60° \\ \to \cos\theta = +1/2}}{\cos\theta}\right) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$\Rightarrow P(a_1 = b_2) + P(a_2 = b_3) + P(b_3 = a_1) = \frac{3}{4} < 1 \ !$$

$\lightning$

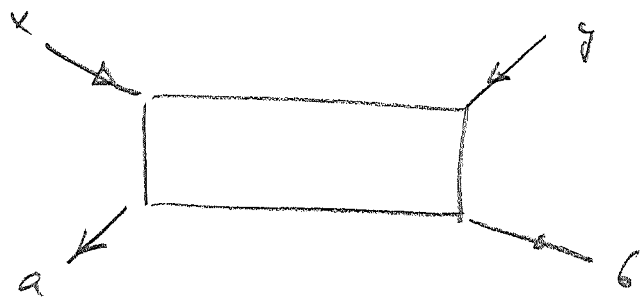
→ Quantum mechanical prediction incompatible with a local realistic description!

→ We cannot assign values to observables we have not measured ("realism").

---

More Bell inequalities:

Formal setup:




x, y: measurement direction (input)

a, b: values of observables (output)

Which output distributions $P(a,b|x,y)$ are consistent with a given physical theory?

## Local hidden variable (LHV) models – local realism:

All outcomes are given through some "hidden" variable which is set beforehand (no faster-than-light):

$$P(a,b|x,y) = \sum_\lambda p_\lambda \, P^A_\lambda(a|x) \, P^B_\lambda(b|y)$$

hidden variable

prob. over $\lambda$

can be deterministic (encode in $\lambda$)

Consider now $x = 0,1$ and $y = 0,1$, with outcomes (measurements) $a_0, a_1, b_0, b_1 = \pm 1$.

Since $a_i = \pm 1$, $b_i = \pm 1$:

$$C = (a_0 + a_1) b_0 + (a_0 - a_1) b_1 = \pm 2$$

$$\Rightarrow \; |\langle C \rangle| \leq \langle |C| \rangle = 2$$

avg. over P

CHSH equality (Clauser, Horne, Shimony, Holt):

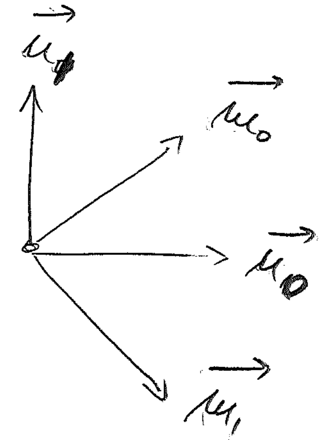
$$|\langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle| \le 2$$

Quantum mechanics:

Use $|\psi^-\rangle$;

$a_i = \vec{\sigma}^A \cdot \vec{u}_i$

$b_i = \vec{\sigma}^B \cdot \vec{u}_i$



$$\langle a_i b_j\rangle = -\cos\theta$$

$$\langle a_0 b_0\rangle = \langle a_1 b_0\rangle = \langle a_0 b_1\rangle = \cos 45° = \frac{1}{\sqrt{2}}$$

$$\langle a_1 b_1\rangle = \cos 135° = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow |\langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle b_0 a_1\rangle - \langle a_1 b_1\rangle| = 2\sqrt{2} > 2.$$

→ Incompatible w/ LHV models.

→ Note: Unlike original Bell inequality, does not require knowledge of eg. correlations in same dirs.