Lecture 7

(I) Protocols

Noiseless qubit channel
Noiseless classical bit channel
Noiseless entanglement

} nonlocal unit resource

<u>Nonlocal</u>: two spatially separated parties share it or if one party uses it to communicate to another

<u>Unit resource</u>: if it comes in some "gold standard" form, such as qubits, classical bits or entangled bits (ebits)

- Noiseless qubit channel: any mechanism that implements the following map

$$|i\rangle_A^* \rightarrow |i\rangle_B \quad (\text{i.e. } |\varphi_i\rangle_A \rightarrow |\psi_i\rangle_B),$$

where $i \in \{0, 1\}$, $\{|0\rangle_A, |1\rangle_A\}$ is some orthonormal basis on Alice's system

do not have to be the same

$\{|0\rangle_B, |1\rangle_B\}$ is some orthonormal basis on Bob's system

The above map is linear and preserves superposition states

$$\alpha|0\rangle_A + \beta|1\rangle_A \rightarrow \alpha|0\rangle_B + \beta|1\rangle_B.$$

Noiseless qubit channel can be written as:

$$\sum_{i=0}^{1} |i\rangle^B \langle i|^A.$$

We label the communication resource of a noiseless qubit channel as follows:

$[q \to q]$ – one forward use of a noiseless qubit channel.

- A noiseless classical bit channel: any mechanism that implements the following map:

$$|i\rangle\langle i|_A \to |i\rangle\langle i|_B$$

$$|i\rangle\langle j|_A \to 0 \quad \text{for } i \neq j,$$

where $i, j \in \{0, 1\}$ and orthonormal bases are again arbitrary. We can write it as:

$$\rho \to \sum_{i=0}^{1} |i\rangle^B \langle i|^A \rho |i\rangle^A \langle i|^B$$

This resource is weaker than noiseless qubit channel, since it does not preserve superposition states.

We denote the communication resource of a noiseless classical bit channel as:

$[c \to c]$ – one forward use of a noiseless classical bit channel.

It is possible for a noiseless qubit channel to simulate a noiseless classical bit channel and we denote this fact with the following resource inequality:

$$[q \to q] \geq [c \to c].$$

- Shared entanglement resource.

The "ebit" is our "gold standard" resource for pure bipartite ~~en~~ (two-party) entanglement.

An ebit is the following Bell state:

$$|\Phi^+\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}},$$

where Alice posses ~~one~~ the first qubit and Bob ~~does~~ posses the second. The resource is denoted as [qq].

(I.1) Entanglement Distribution

We show how a noiseless qubit channel can generate a noiseless ebit. The protocol consists of two steps:

1. Alice prepares a Bell state locally in her lab: she first prepares two qubits
   $$|0\rangle^A |0\rangle^{A'}$$
   and then performs a Hadamard gate on qubit A:
   $$\left(\frac{|0\rangle^A + |1\rangle^A}{\sqrt{2}}\right) |0\rangle^{A'}.$$
   She then performs a CNOT gate with qubit A as the source and qubit A' as the target. The state becomes
   $$\frac{|00\rangle^{AA'} + |11\rangle^{AA'}}{\sqrt{2}} = |\Phi^+\rangle^{AA'}$$

2. Alice sends qubit A' to Bob with one use of noiseless qubit channel. Alice & Bob share the ebit $|\Phi^+\rangle_{AB}$.

The resource inequality of this protocol is

$$[q \to q] \geq [qq]$$

Notes: notice the difference between

Bell state – local state is Alice's lab

and

ebit – a nonlocal resource shared between Alice and Bob.

(I. 2) Quantum Super-Dense Coding

We know that with one use of noiseless quantum channel we can transmit one classical bit.

Super-dense coding doubles classical bits by using noiseless entanglement.

1. Suppose Alice and Bob share an ebit $|\Phi^+\rangle_{AB}$.
   Alice applies one of four unitary operations $\{I, X, Z, XZ\}$ to her side of the above state. The state becomes one of the four Bell states, depending on the message that Alice chooses:

   $$|\Phi^+\rangle_{AB}, \ |\Phi^-\rangle_{AB}, \ |\Psi^+\rangle_{AB}, \ |\Psi^-\rangle_{AB}$$

2. Alice transmits her qubit to Bob with one use of noiseless qubit channel

3. Bob performs a Bell measurement (a meas. in the basis $\{\Phi^+, \Phi^-, \Psi^+, \Psi^-\}$ to distinguish the four states.

Thus Alice can transmit 2 class. bits (corresponding to 4 mess.) if she uses a noiseless q.ch. and shares an ebit with Bob.

The super-dense coding protocol implements the following resource inequality:

$$[qq] + [q \to q] \geq 2\,[c \to c].$$

I.3 Quantum Teleportation

The protocol destroys the quantum state of a qubit in one location and recreates it on a qubit at a distant location, with the help of shared entanglement.

Algebraic calculations in preparation for the protocol:

Consider a qubit $|\psi\rangle^{A'}$ that Alice possesses, where

$$|\psi\rangle_{A'} = \alpha|0\rangle_{A'} + \beta|1\rangle_{A'} .$$

Suppose Alice also shares a maximally entangled state $|\Phi^+\rangle_{AB}$ with Bob. The joint state of the systems A, A', B is as follows: $|\psi\rangle_{A'} |\Phi^+\rangle_{AB}$

$$|\psi\rangle_{A'} |\Phi^+\rangle_{AB} = \left(\alpha|0\rangle_{A'} + \beta|1\rangle_{A'}\right)\left(\frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left[\alpha|000\rangle_{A'AB} + \beta|100\rangle_{A'AB} + \alpha|011\rangle_{A'AB} + \beta|111\rangle_{A'AB}\right]$$

use Bell states on system $A'A$:

$$|00\rangle_{A'A} = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle_{A'A} + |\Phi^-\rangle_{A'A}\right)$$

$$|10\rangle_{A'A} = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle_{A'A} - |\Psi^-\rangle_{A'A}\right)$$

$$|01\rangle = \dots$$

$$|11\rangle = \dots$$

$$= \frac{1}{2}\Big[\alpha\left(|\Phi^+\rangle + |\Phi^-\rangle\right)_{A'A}|0\rangle_B + \beta\left(|\Psi^+\rangle - |\Psi^-\rangle\right)_{A'A}|0\rangle_B$$

$$+ \alpha\left(|\Psi^+\rangle + |\Psi^-\rangle\right)_{A'A}|1\rangle_B + \beta\left(|\Phi^+\rangle - |\Phi^-\rangle\right)_{A'A}|1\rangle_B\Big]$$

$$= \frac{1}{2}\Big[|\Phi^+\rangle_{A'A} (\alpha|0\rangle + \beta|1\rangle)_B + |\Phi^-\rangle_{A'A} (\alpha|0\rangle - \beta|1\rangle)_B$$

$$+ |\Psi^+\rangle_{A'A} (\alpha|1\rangle + \beta|0\rangle)_B + |\Psi^-\rangle_{A'A} (\alpha|1\rangle - \beta|0\rangle)_B \Big]$$

using Pauli matrices X, Z and their action on $|\psi\rangle$:

$$X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$$

$$Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$$

$$XZ|\psi\rangle = \alpha|1\rangle - \beta|0\rangle$$

$$= \frac{1}{2}\Big[|\Phi^+\rangle_{A'A} |\psi\rangle_B + |\Phi^-\rangle_{A'A} Z|\psi\rangle_B + |\Psi^+\rangle_{A'A} X|\psi\rangle_B + |\Psi^-\rangle_{A'A} XZ|\psi\rangle_B \Big].$$

Quantum teleportation protocol:

1. Alice possesses a qubit $|\psi\rangle_A$, and shares an ebit with Bob. She performs a Bell measurement on system $A'A$. The state collapses to one of the following four states with uniform probability:

$$|\Phi^+\rangle_{A'A} |\psi\rangle_B$$

$$|\Phi^-\rangle_{A'A} Z|\psi\rangle_B$$

$$|\Psi^+\rangle_{A'A} X|\psi\rangle_B$$

$$|\Psi^-\rangle_{A'A} XZ|\psi\rangle_B$$

2. Notice that the state is a product state with respect to the cut $A'A - B$. At this point Alice already knows what state Bob has, because she knows the result of the measurement. On the other hand, Bob doesn't know anything about his state.

2. Alice transmits two classical bits to Bob that indicate which of the four measurement results she obtains.
Now Bob knows which operation he needs to perform in order to restore his state to Alice's original $|\psi\rangle$. ~~[illegible]~~

3. Bob performs the restoration operation: I, X, Z, XZ

The resource inequality for q. teleportation is as follows:

$$[qq] + 2[c \to c] \geq [q \to q].$$

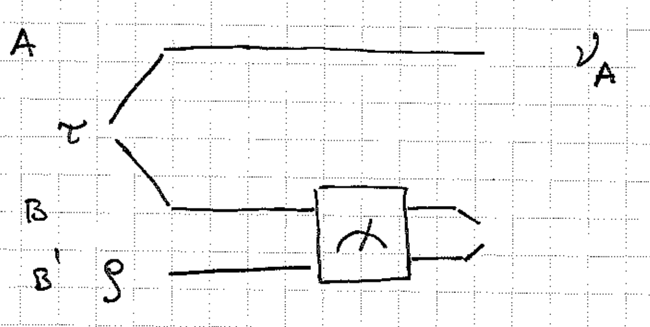
II <u>Implementation of Choi-Jamiolkowski iso via teleportation</u>

If T is a quantum channel then its Jam. state τ can operationally be obtained by letting T act on a max. entangled state.
Converse? Given τ, how to implement T as an action on any state ϱ?

1. Alice & Bob share state $\tau = (T_A \otimes \mathbb{1}_B)|\Omega\rangle\langle\Omega|$, where $|\Omega\rangle = \sum_i \frac{1}{\sqrt{d}} |ii\rangle_{AB}$.

2. Bob also has a state ϱ (on system B').
He performs a measurement on his system BB' using a POVM which contains state $\omega = |\Omega\rangle\langle\Omega|$.

3. Alice's state after the measurement is $T(\varrho)$ if Bob has obtained a meas. outcome corresp. to ω.



Denote Alice's state by ν_A after a succesful Bob's measurement, which occurs with prob. p.

$$\nu_A = \frac{1}{p} \, \mathrm{Tr}_{BB'} \left((\mathbb{1}_A \otimes \omega_{BB'})(\tau_{AB} \otimes \rho_{B'})(\mathbb{1}_A \otimes \omega_{BB'}) \right)$$

To show that $\nu_A = T(\rho)$ compute the exp. value for any A:

$$p \, \mathrm{Tr}(A \nu_A) = \mathrm{Tr}\left((A_A \otimes \mathbb{1}_{BB'})(\mathbb{1}_A \otimes \omega_{BB'})(\tau_{AB} \otimes \rho_{B'})(\mathbb{1}_A \otimes \omega_{BB'}) \right)$$

$$= \mathrm{Tr}\left((\tau_{AB} \otimes \rho_{B'})(A_A \otimes \omega_{BB'}) \right) \ominus$$

Remind that $\omega = \frac{1}{d} \sum_{ij} |i\rangle\langle j|_B \otimes |i\rangle\langle j|_{B'}$.

Denote the coeff. $\rho = \sum_{kl} \lambda_{kl} |k\rangle\langle l|$ in the above basis.

$$\ominus \; \mathrm{Tr}\left(\left(\tau_{AB} \otimes \sum_{kl} \lambda_{kl} |k\rangle\langle l|_{B'}\right) \left(A \otimes \frac{1}{d} \sum_{ij} |i\rangle\langle j|_B \otimes |i\rangle\langle j|_{B'} \right) \right)$$

$$= \frac{1}{d} \sum_{\substack{kl \\ ij}} \mathrm{Tr}\left(\tau (A_A \otimes |i\rangle\langle j|_B) \otimes \lambda_{kl} |k\rangle\langle l|i\rangle\langle j|_{B'} \right)$$

$$= \begin{cases} l = i \\ k = j \text{ – taking a trace over } B' \end{cases}$$

$$= \frac{1}{d} \mathrm{Tr}\Big(\tau \big(A \otimes \underbrace{\sum_{ij} |i\rangle\langle j| \lambda_{ji}}_{\rho^T} \big) \Big) = \mathrm{Tr}(\tau \, A \otimes \rho^T) \frac{1}{d}$$

$$= \frac{1}{d^2} \mathrm{Tr}(A \, T(\rho))$$

Therefore $\nu_A = T(\rho)$ and $p = \frac{1}{d^2}$.