III.5 Mixed state entanglement

a) Introduction

When is a mixed state entangled?

i) If ρ_{AB} cannot be created by LOCC!

ii) If we can extract ebits from it.

iii) If it helps us do sth. better (w/ LOCC).

Use i)

States which can be prepared by LOCC:

$$(*) \quad \boxed{\rho = \sum p_i \, \rho_i^A \otimes \rho_i^B \quad \text{"separable state"}}$$

$$(\rho_i^A, \rho_i^B \geq 0, \; p_i \geq 0)$$

ρ entangled $:\Leftrightarrow$ ρ not separable (= cannot be written as $(*)$)

Given ρ, how can we test if it is entangled?

Problem: Given ρ, unclear how to find sep. decomposition

$$\rho = \sum p_i \, \rho_i^A \otimes \rho_i^B$$

($\rightarrow$ ambiguity of ensemble interpret.: need to optimize over isometries!)

6) Entanglement witnesses

Structure of sep. states:

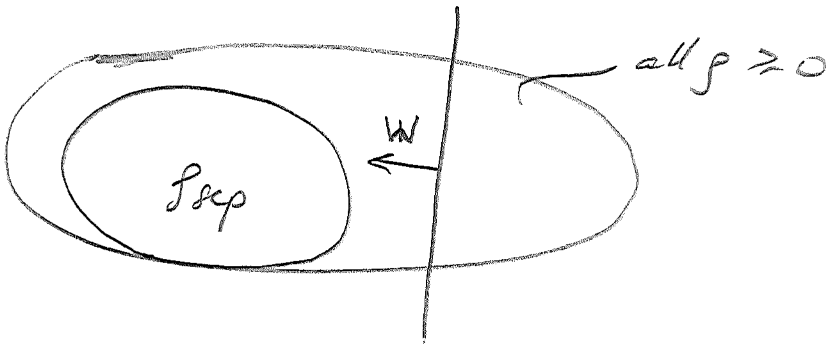
Let $\rho = \sum p_i \rho_i^A \otimes \rho_i^B$; $\sigma = \sum q_j \sigma_j^A \otimes \sigma_j^B$

$$\Longrightarrow \lambda\rho + (1-\lambda)\sigma = \sum r_k \chi_k^A \otimes \chi_k^B \qquad \lambda \in [0;1]$$

with $r_k = (\lambda p_1, \lambda p_2, \dots, (1-\lambda) q_1, \dots)$;

$\chi_k^{A/B} = (\rho_1^{A/B}, \rho_2^{A/B}, \dots, \sigma_1^{A/B}, \dots)$

$\Longrightarrow \lambda\rho + (1-\lambda)\sigma$ separable: sep. states form <u>convex set</u>



Can find hyperplanes s.th. all ρ on one side are entangled.

Characterize plane + direction by $\overbrace{\text{vector}}^{\text{normal}}$ $W = W^+$:

$$\rho \text{ sep} \Longrightarrow tr(W\rho) \geq 0$$

i.e.: $tr(W\rho) < 0 \Longrightarrow$ <u>ρ entangled!</u>

<u>W: "entanglement witness"</u>

Notes:

- Need to make sure $tr(W\rho_{sep}) \geq 0$!
- Witness only detects <u>certain</u> ent. states!
- Convex set $\equiv$ all tangent planes:

  $\Rightarrow \exists$ witness for <u>any</u> ent. states.

- Witness <u>linear operator</u> $\Rightarrow$ experimentally measurable (in part. if W is simple)

<u>Example</u>:

$$W = \mathbb{F} \text{ ("flip")}; \quad \mathbb{F} = \sum_{i,j=1}^{d} |ij\rangle\langle ji|$$

$$\rho_{sep} = \sum p_i \rho_i^A \otimes \rho_i^B :$$

$$tr(W\rho_{sep}) = \sum p_i \, tr(\mathbb{F} \cdot \rho_i^A \otimes \rho_i^B) \overset{(1)}{=}$$

$$= \sum p_i \underbrace{tr(\rho_i^A \rho_i^B)}_{\overset{(2)}{\geq} 0} \geq 0$$

$\boxed{\begin{array}{l}\text{Use:} \\ (1)\ tr[\mathbb{F} \cdot (A \otimes B)] = tr(AB) \\ (2)\ P, Q \geq 0 \Rightarrow tr(PQ) \geq 0\end{array}}$

Which states does it detect? $\longrightarrow$ Those w/ prevalent anti-sym. component!

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \Rightarrow \mathbb{F}|\psi^-\rangle = -|\psi^-\rangle \Rightarrow \langle\psi^-|\mathbb{F}|\psi^-\rangle = -1.$$

Mixed states:

$$\rho = \lambda |\psi^-\rangle\langle\psi^-| + (1-\lambda)\frac{\mathbb{1}}{4} \quad ; \lambda \in [-\tfrac{1}{3}, 1] : \quad (\text{"Werner state"})$$

$$\mathrm{tr}[\mathbb{F}\rho] = \lambda \underbrace{\langle\psi^-|\mathbb{F}|\psi^-\rangle}_{-1} + (1-\lambda)\underbrace{\mathrm{tr}\left[\frac{\mathbb{1}}{4}\mathbb{F}\right]}_{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{3}{2}\lambda$$

$\Rightarrow$ state ent. if $\lambda \geq \frac{1}{3}$.

What about $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$?

$\mathbb{F}|\phi^+\rangle = |\phi^+\rangle \Rightarrow \langle\phi^+|\mathbb{F}|\phi^+\rangle = 1 \Rightarrow$ not detected!

Optimal? Yes. E.g. $\rho = |0\rangle\langle 0| \otimes |1\rangle\langle 1| : \mathrm{tr}(\mathbb{F}\rho) = 0$.

Other witnesses: E.g. $W = \mathbb{1} - d \cdot |\Omega\rangle\langle\Omega|$; $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |d,d\rangle$

$\Longrightarrow$ Homework

c) Positive maps and the PPT criterion

Reminder: $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ positive $:\Leftrightarrow (\rho \geq 0 \Rightarrow \Lambda(\rho) \geq 0)$

Usually: require Λ completely positive (CP)

Here: Will be interested in Λ positive but not CP.

Consider $\rho_{sep} = \sum_i p_i \rho_i^A \otimes \rho_i^B$:

$$(\Lambda \otimes I)(\rho_{sep}) = \sum_i p_i \underbrace{\Lambda(\rho_i^A)}_{=\tilde{\rho}_i^A \geq 0} \otimes \rho_i^B = \tilde{\rho}_{sep} \geq 0$$

i.e.: $(\Lambda \otimes I)(\rho) \ngeq 0 \Rightarrow$ ρ entangled!

has neg. eigenvalues

Most important example:

$$\Lambda(\rho) = \rho^T$$

$(\Lambda \otimes I) =: \rho^{T_A}$: "partial transpose"

(i.e.: $\rho = \sum \rho_{ij}^{i'j'} |i,j\rangle\langle i',j'| \Rightarrow \rho^{T_A} = \sum \rho_{ij}^{i'j'} |i',j\rangle\langle i,j'|$)

i.e.: $\boxed{\begin{array}{l} \rho^{T_A} \ngeq 0 \Rightarrow \rho \text{ entangled} \\ \text{positive partial transpose (PPT) criterion} \end{array}}$

E.g.: $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle$

$$\Rightarrow (|\Omega\rangle\langle\Omega|)^{T_A} = \frac{1}{d} \sum (|i,i\rangle\langle j,j|)^{T_A} = \frac{1}{d} \sum |j,i\rangle\langle i,j| .$$

Not positive: e.g. $|v\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$:

$$\langle v| \left[\left(|\Omega\rangle\langle\Omega| \right)^{T_A} |v\rangle \right] = \frac{1}{2}\left(\langle 01| - \langle 10|\right)\left(|10\rangle - |01\rangle\right) = -1$$

Again: Also works for $\rho = \lambda |\Omega\rangle\langle\Omega| + (1-\lambda)\frac{\mathbb{I}}{d^2}$ ("isotropic state")

E.g. $d=2$: $$\rho = \begin{pmatrix} \frac{\lambda}{2} & 0 & 0 & \frac{\lambda}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\lambda}{2} & 0 & 0 & \frac{\lambda}{2} \end{pmatrix} + \frac{1-\lambda}{4} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+\lambda}{4} & & & \frac{\lambda}{2} \\ & \frac{1-\lambda}{4} & & \\ & & \frac{1-\lambda}{4} & \\ \frac{\lambda}{2} & & & \frac{1+\lambda}{4} \end{pmatrix}$$

$$\Rightarrow \rho^{T_A} = \begin{pmatrix} \frac{1+\lambda}{4} & & & \\ & \frac{1-\lambda}{4} & \frac{\lambda}{2} & \\ & \frac{\lambda}{2} & \frac{1-\lambda}{4} & \\ & & & \frac{1+\lambda}{4} \end{pmatrix}$$

$\Rightarrow$ positive iff $\frac{\lambda}{2} \leq \frac{1-\lambda}{4} \Longleftrightarrow \underline{\underline{\lambda \leq \frac{1}{3}}}$

Notes: Indep. of unitary on B $\Rightarrow$ detects all ME states!

(i.e.: Stronger than witness!)

In fact: PPT criterion detects all entangled states in dimension $d_A \times d_B = 2\times 2$ and 3×2 (but not 3×3 or 4×2 $\Rightarrow$ "PPT bound ent. states")

Other example:

$$\Lambda(\rho) = \mathrm{tr}(\rho) I - \rho :$$

$$(\Lambda \otimes I)(\rho) = \underbrace{\mathrm{tr}\,\rho}_{=1} \cdot (I \otimes \mathrm{tr}_A \rho) - \rho = I \otimes \rho_B - \rho \not\geq 0 \Rightarrow \text{entangled}$$

"reduction criterion" $I \otimes \mathrm{tr}_A \rho \not\geq \rho$ $\Rightarrow$ Homework.

d) Relation of witnesses & positive maps:

For each witness W, there is a pos. map Λ which detects at least as good as W (in fact, better):

Witness = bipartite "state" (rather: operator) W

map $\cong$ Jamiolkowski map of W^T,

$$\Lambda(X) = \mathrm{tr}_A\left(W^T (X_A^T \otimes I_B)\right) = \mathrm{tr}_A\left(W(X_A \otimes I_B)\right)^T$$

Then: $\mathrm{tr}(W(A \otimes B)) = \mathrm{tr}_B\big(\underbrace{\mathrm{tr}_A(W(A \otimes I))}_{= \Lambda(A)^T} \cdot B\big) = \mathrm{tr}(\Lambda(A)^T B)$

$$= \sum_{ij} [\Lambda(A)^T]_{ij} B_{ji} = d \langle \Omega | \underbrace{\Lambda(A) \otimes B}_{= (\Lambda \otimes I)(A \otimes B)} | \Omega \rangle$$

linearity $\Rightarrow$ $\mathrm{tr}(W\rho) = d \langle \Omega | (\Lambda \otimes I)(\rho) | \Omega \rangle$.

i.e.: $\mathrm{tr}(W\rho) < 0 \Rightarrow (\Lambda \otimes I)(\rho) \not\geq 0$.

$\Rightarrow$ Λ stronger than W!

e.g. $W = \mathbb{F}$:

$$\Lambda(X) = tr_B(\mathbb{F}(I_A \otimes X_B))^T = tr(I_A \cdot X_B)^T = X_B^T.$$

$\Rightarrow$ PPT criterion!

<u>Note:</u> PPT <u>strictly</u> stronger: $\mathbb{F}$ could not detect e.g. $|\Omega\rangle$!

<u>Corollary:</u> A state is entangled if & only if
$(\Lambda \otimes I)(\rho) \geq 0$ $\forall$ positive Λ (as sep. states $\Leftrightarrow$ all witnesses)

e) <u>Quantification of mixed state entanglement</u>

How to quantify entanglement?

i) Entanglement needed to <u>create</u> state

"Entanglement of formation" E_F (single copy)

"Entanglement cost" E_C (many copies)

ii) Extractable entanglement:

"Distillable entanglement" E_D:

LOCC protocol $\mathcal{E}_n$: $\| \mathcal{E}_n(\rho^{\otimes n}) - |\Omega\rangle\langle\Omega|^{\otimes m} \| \to 0.$

<u>Note:</u> Generally $E_C \gneq E_D$: no unique measure!