V. Quantum error correction

1) Introduction

- Coupling to environment $\longrightarrow$ errors
- Class. systems: macroscopic $\longrightarrow$ errors unlikely
- Quantum comp: – need single qubits (= quantum system)

  $\longrightarrow$ fragile!

  – need coupling to realize gates!

Can we protect q. information from noise?

Classical error correction:

Copy information.

e.g.: assume indep. bit flip errors w. prob. p.

encode 1 bit in 3:

$$0 \longrightarrow \hat{0} = 000$$

$$1 \longrightarrow \hat{1} = 111$$

Correction: majority vote:

$$000,\ 001,\ 010,\ 100 \xrightarrow{\text{correct to}} 000$$

$$111,\ 110,\ 101,\ 011 \longrightarrow 111$$

$$P_{error} = \text{prob}(\geq 2 \text{ flips}) = p^3 + 3p^2(1-p) \leq 3p^2 < p$$

for $p < 1/3$!

$\Rightarrow$ effective error prob decreased!

To obtain better robustness:

- concatenate codes
- use more bits
- use smarter codes

Quantum error correction:

Several problems:

- cannot clone qubits (and, if we could, we couldn't compare them!)
- different types of errors exist, e.g. X (bit flip) and Z (phase flip)
- errors can be continuous
- measuring qubits destroys quantum info

a) The 3-qubit bit flip code

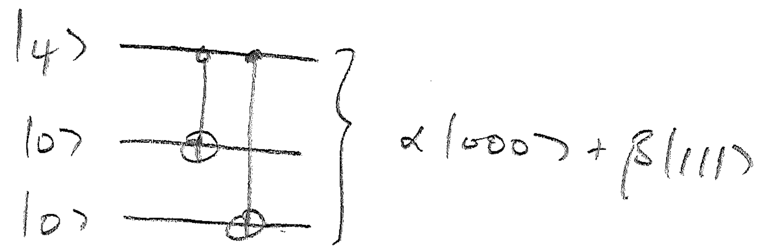
Try to copy in fixed (comput.) basis:

$$|0\rangle \longmapsto |\hat{0}\rangle = |000\rangle$$

$$|1\rangle \longmapsto |\hat{1}\rangle = |111\rangle$$

i.e.: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xmapsto{\text{encoding}} \alpha|000\rangle + \beta|111\rangle$

Encoding circuit:



Consider bit flip error $|\phi\rangle \longmapsto X|\phi\rangle$.

Want: Correction procedure which can correct one bit flip error.

But: Meas. all qubits would destroy quantum info!

$\Longrightarrow$ Need to measure only info about error (i.e., which bit has been flipped)!

POVM elements ("syndrome measurement")

no flip: $P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$

1st qubit flipped: $P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$

2nd —"—: $P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$

3rd —"—: $P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$

→ only 2 bits of information acquired!
(namely the error syndrome)

E.g. consider

$$\alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{qubit } 1]{\text{bit flip on}} \alpha|100\rangle + \beta|011\rangle.$$

⇒ meas. result P_1, post-meas. state

$$\alpha|100\rangle + \beta|011\rangle$$

⇒ recovery operation: flip qubit 1: X_1 → $\alpha|000\rangle + \beta|111\rangle$.

Note: Since this works for any state, it also works for parts of an entangled state (b/c linearity!)

$$\alpha|0\rangle|a\rangle + \beta|1\rangle|b\rangle \xrightarrow{\text{encode}} \alpha|000\rangle|a\rangle + \beta|111\rangle|b\rangle$$

$$\xrightarrow{\text{error } X_1} \alpha|100\rangle|a\rangle + \beta|011\rangle|b\rangle \xrightarrow[\text{correct: } X_1]{\text{meas: } P_1} \alpha|000\rangle|a\rangle + \beta|111\rangle|b\rangle.$$

What about arbitrary (continuous) errors, e.g.

$$|\phi\rangle \longmapsto e^{i\vartheta X}|\phi\rangle = (\cos\vartheta\, \mathbb{1} + i\sin\vartheta\, X)|\phi\rangle \,?$$

$$\alpha|000\rangle + \beta|111\rangle \xmapsto{\text{error}} \cos\theta\,(\alpha|000\rangle + \beta|111\rangle) + i\sin\theta\,(\alpha|100\rangle + \beta|011\rangle)$$

syndrome meas.: collapse onto:

$p = \cos^2\vartheta$: $P_0 \Rightarrow \alpha|000\rangle + \beta|111\rangle$: no corr. necessary ✓

$p = \sin^2\vartheta$: $P_1 \Rightarrow \alpha|100\rangle + \beta|011\rangle \xrightarrow[X_1]{\text{correction:}}$ ✓

Meas. of error syndrome P_i: collapses error onto

no error or bit flip error:

Cont. error is mapped to discrete error ("digital error").

Different perspective on syndrome meas. + correction:

$|000\rangle, |111\rangle$: $+1$-eigenstates of Z_1Z_2 & Z_2Z_3.
("stabilizer")

Measure Z_1Z_2 & Z_2Z_3 (i.e., compare $1\leftrightarrow 2$ & $2\leftrightarrow 3$):

$(+1,+1)$: no error

$(-1,+1)$: error on qubit 1

$(+1,-1)$: —— " —— 3

$(-1,-1)$: —— " —— 2

More formally:

X_1 anti-com. w $Z_1 Z_2$:

$|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$, $\quad Z_1 Z_2 |\psi\rangle = |\psi\rangle$:

$$\langle\psi| X Z_1 Z_2 X |\psi\rangle = -\langle\psi| Z_1 Z_2 |\psi\rangle = -1.$$

$\Longrightarrow (-1)$: error anti-comm. w/ $Z_1 Z_2$ occurred.

Correction operator $\longleftrightarrow$ satisfies same anti-comm. relations!

... more on this later!

5) 3-qubit phase flip code

What about Z errors?

$$\alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{one qubit}]{Z\text{ error on}} \alpha|000\rangle - \beta|111\rangle.$$

$\rightarrow$ still in code space (i.e., valid state) $\Longrightarrow$ error not detectable!

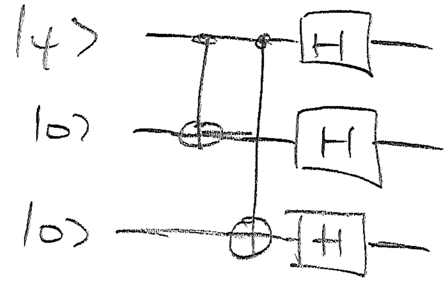
(cf.: error comm. w. $Z_1 Z_2$ & $Z_2 Z_3$)

But: $Z|+\rangle = |-\rangle$; $Z|-\rangle = |+\rangle$

$\longleftrightarrow$ $Z \mathrel{\hat{=}}$ bit flip error in $|\pm\rangle$ basis.

Encoding $|\hat{0}\rangle = |{+}{+}{+}\rangle$, $|\hat{1}\rangle = |{-}{-}{-}\rangle$ will

protect against Z error!



Syndrome measurements: $H^{\otimes 3} P_k H^{\otimes 3}$, or via

stabilizers $X_1 X_2$ and $X_2 X_3$.

<u>Recovery</u>: $H X_i H = Z_i$ (anti-com. w. stabilizers).

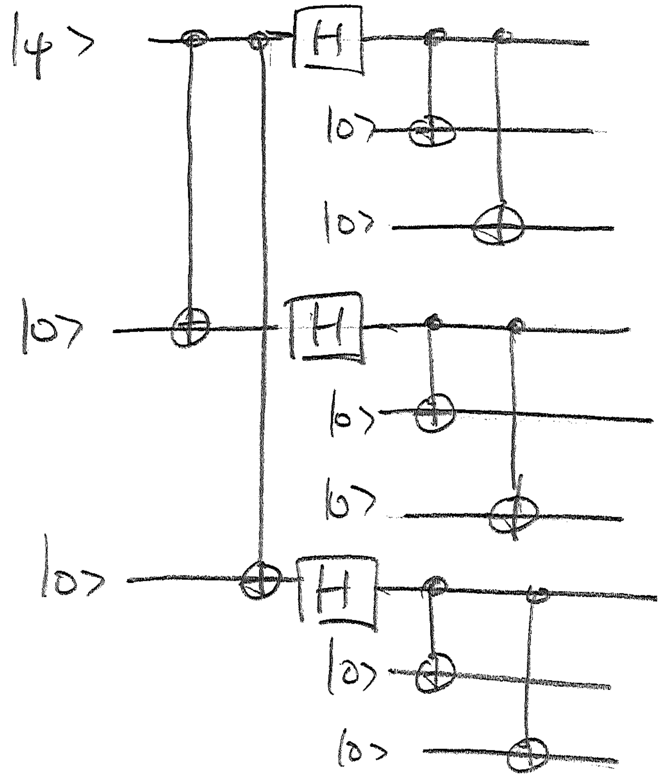
<u>Problem</u>: Now no protection against bit flip (X) errors!

2. The 9-qubit Shor code

<u>Solution</u>: Concatenate 3-qubit bit flip w/ 3-qubit phase flip code!

$$|0\rangle \longmapsto |+\rangle|+\rangle|+\rangle \longmapsto \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \longmapsto |-\rangle|-\rangle|-\rangle \longmapsto \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{2\sqrt{2}}$$



9-qubit Shor code

Shor code protects against <u>arbitrary single-qubit errors</u>.

Focus on X, Z, and $XZ \hat{=} Y$ errors (all err. collapse to those)

<u>Syndrome meas. + correction:</u>

<u>Single bit flip (X error):</u>

meas. $Z_1 Z_2$, $Z_2 Z_3$

$Z_4 Z_5$, $Z_5 Z_6$

$Z_7 Z_8$, $Z_8 Z_9$

<u>E.g.:</u> Bit flip X_1 : $\langle Z_1 Z_2 \rangle = -1$, rest $= +1$

(also follows from comm. relations).

Correction: Apply corresp. X_i (anti-comm. w/ all stabilizers S w/ $\langle S \rangle = -1$).

Single phase flip (Z error):

phase flip: $|000\rangle \pm |111\rangle \longleftrightarrow |000\rangle \mp |111\rangle$.

$\Rightarrow$ syndrome = compare phase of adjacent encoded blocks.

"$X_{123} X_{456}$" and "$X_{456} X_{789}$":

Stabilizers are

$$X_1 X_2 X_3 X_4 X_5 X_6$$
$$X_4 X_5 X_6 X_7 X_8 X_9$$

(Note: These act as "$X_{123} X_{456}$" & "$X_{456} X_{789}$" on the encoded qubits.)

$\Rightarrow$ correction by any anti-comm. operation!

E.g.: Z error on 4: $|000\rangle + |111\rangle \longleftrightarrow |000\rangle - |111\rangle$

$$\left.\begin{array}{l} \langle X_1 X_2 X_3 X_4 X_5 X_6 \rangle = -1 \\ \langle X_4 X_5 X_6 X_7 X_8 X_9 \rangle = -1 \end{array}\right\} \text{ Z error on 4, 5, or 6}$$

Correction: Z_4, Z_5, Z_6, or $Z_4 Z_5 Z_6$.

Note: All X & Z stabilizers commute;

- 9 qubits, 8 indep. stabilizers $\Rightarrow$ fix 2-dim qubit space!
- can be measured simultaneously
- correction must anti-com. w/ all stabilizers w/ $\langle S \rangle = -1$, and commute w/ all w/ $\langle S \rangle = 1$.

9-qubit code also protects against Y errors:

E.g.: $Y_2 \propto Z_2 X_2$

anti-comm. w/ $X_1 X_2 X_3 X_4 X_5 X_6$

—— " —— $Z_1 Z_2$ & $Z_2 Z_3$

$\Rightarrow$ correction $Z_2 X_2$, or $Z_1 Z_2 Z_3 X_2$.

$\Rightarrow$ error fully corrected (up to global phase).

In fact: protection against arb. errors

$$e^{i\vartheta \vec{n}\cdot\vec{\sigma}} = \cos\vartheta\, \mathbb{1} + i \sin\vartheta\, \vec{n}\cdot\vec{\sigma}$$

$\Rightarrow$ syndrome meas. projects onto (anti-)comm. error, i.e., Pauli error X, Y, or Z.

What if errors occur on <u>more than one qubit</u>?

Some – but not all – errors are corrected:

<u>E.g.:</u> $X_1 X_4$: correctable.

$Z_1 Z_2$: trivial (no error)

<u>But:</u> $X_1 X_2$: breaks code $\lightning$

$Z_1 Z_4$: breaks code $\lightning$

$\Rightarrow$ <u>Concatenate codes</u> or <u>use better codes.</u>