Lecture "Analytical and Numerical Methods for Quantum Many-Body Systems from a Quantum Information Perspective" — Exercise Sheet #4

1. Show that the AKLT Hamiltonian, i.e., the projector onto the spin 2 subspace of two spin 1 particles (labelled 1 and 2), can be written as $\frac{1}{2}\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{6}(\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{1}{3}$.

   (You can do this "brute force", but it might be easier – and is certainly more elegant and insightful – to expand $(\vec{S}_1 + \vec{S}_2)^2$ and check the possible eigenvalues.)

2. Verify the stability of the injectivity condition under blocking – i.e., given an MPS with injective tensors, show that the MPS obtained by blocking tensors still has the injectivity property.

   (There are several ways to do this: You can argue in the "applying a map to entangled pairs" picture, you can use a graphical tensor network argument as in Lemma 3.2 of http://arxiv.org/abs/1001.3807, or you can use the mathematical definition of injectivity, namely that the map $X \mapsto \sum_i \mathrm{tr}[A^i X]|i\rangle$ is injective.)

3. Verify formally (either using graphical tensor network notation or with equations) that the reduced density matrix ρ_2 of 2 sites of an MPS is supported on the space

   $$S_2 = \Big\{ \sum_{ij} \mathrm{tr}[A^i A^j X]|i,j\rangle \Big| X \in M_{D\times D} \Big\} .$$

   Also, use the result of Problem 2 to show that for an injective MPS, ρ_2 is supported on the *full* space.

4. Verify that the four-body parent Hamiltonian for the AKLT state derived in Lecture 9 (the one for which we can prove uniqueness of the ground state) has the same ground space as the original two-body Hamiltonian. (Note: It is sufficient to check the dimension of the ground space. – Why?)

   (This problem is meant to be done on a computer – you can either do this analytically using a computer algebra system, or numerically.)

5. Find a lower bound on the gap of the AKLT model!

   (For this, you first need to check numerically if the condition $h_i h_{i+1} + h_{i+1} h_i \geq -c(h_i + h_{i+1})$ is satisfied for some $c < \frac{1}{2}$ (if not, you need to block more sites – if you need to do so, pay attention to how you overlap the Hamiltonian terms). This gives a bound on the gap of the blocked Hamiltonian, which you then need to relate to the gap of the original Hamiltonian.)

   Together with the result of Problem 4, this shows that the AKLT model has a unique ground state with a gap above.