The Haldane conjecture

Consider the antiferromagnetic Heisenberg model in 1D:

$$H = \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} .$$

Haldane '83: There is a qualitative difference between integer and half-integer $\vec{S}$:

integer S: H has unique ground state (a spectral gap above ("Haldane gap")

half-int. S: H is gapless.

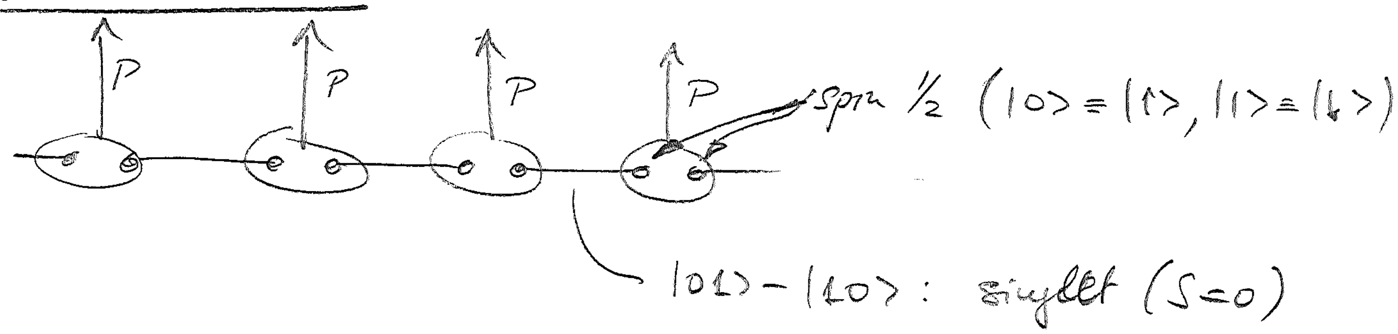
Argument via mapping to field-theoretic models ("non-linear sigma models") for large S – not fully rigorous ("Haldane conjecture")

Note: That H is gapless for half-int. S follows from the Lieb-Schultz-Mattis-Theorem which says that certain rotationally invariant 1D Hamiltonians for half-int. spins cannot have a unique ground state and a gap above.

Q: Can one prove a Haldane gap for int. spin Heis. AFM, or a similar system?

The AKLT model

A. The AKLT State:



Each site consists of two virtual spin-$\frac{1}{2}$ particles. Together, they have spin $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$, i.e., either spin 0 or spin 1.

$P \equiv$ Proj. onto spin-1 space.

In more detail: The four basis states for the two spin-$\frac{1}{2}$ systems:

$$|S=0, m=0\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$$|S=1, m=-1\rangle = |11\rangle$$

$$|S=1, m=0\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|S=1, m=1\rangle = |00\rangle$$

$$P \equiv |-1\rangle\langle 11| + |0\rangle \frac{\langle 01| - \langle 10|}{\sqrt{2}} + |1\rangle\langle 00|$$

The resulting state is $SU(2)$-invariant (i.e., invariant under any rotation generated by the spin algebra S_x, S_y, S_z):

i) P is an "intertwiner", i.e., a change of representations:

$$S_x^{S=1}\, P = P\, \underbrace{\left(S_x^{S=1/2} \otimes \mathbb{1} + \mathbb{1} \otimes S_x^{S=1/2}\right)}_{} \qquad \text{etc.}$$

$$= S_x^{\frac{1}{2}\otimes\frac{1}{2}} \equiv S_x^{S=1} \oplus S_x^{S=0}$$

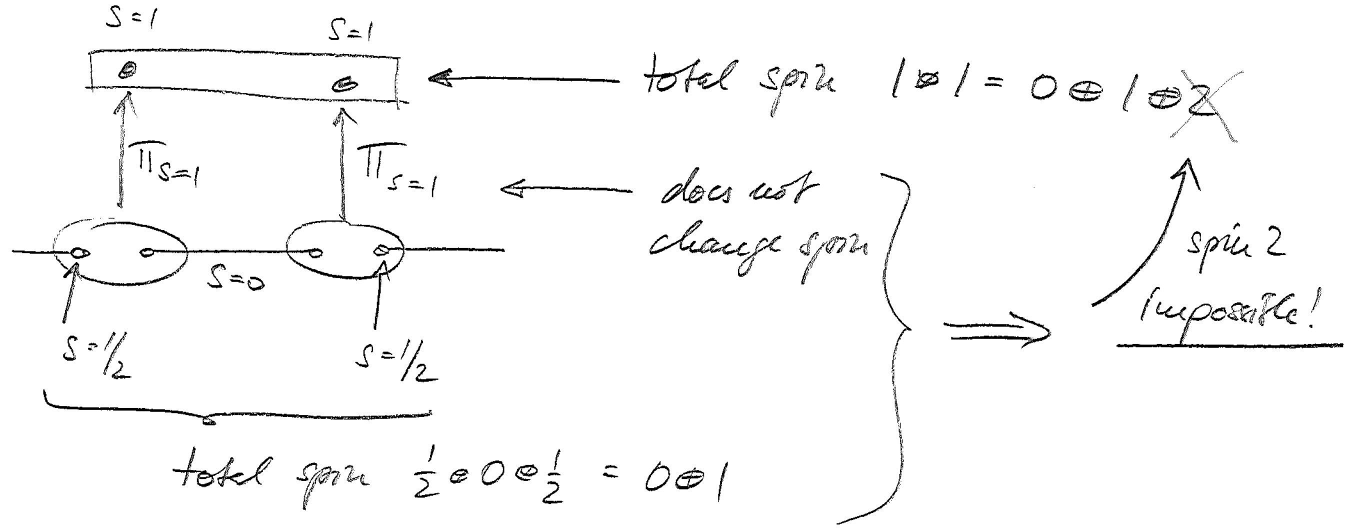
$\Rightarrow$ Any rotation $e^{i\vec{\phi}\cdot\vec{S}}$ equals a rotation on the virtual spins:

$$e^{i\vec{\phi}\cdot\vec{S}^{S=1}}\, P = P\left(e^{i\vec{\phi}\cdot\vec{S}^{S=1/2}} \otimes e^{i\vec{\phi}\cdot\vec{S}^{S=1/2}}\right)$$

ii) The singlet is $SU(2)$-invariant (as it has spin 0):

$$\left(e^{i\vec{\phi}\cdot\vec{S}^{S=1/2}} \otimes e^{i\vec{\phi}\cdot\vec{S}^{S=1/2}}\right)\left(|01\rangle - |10\rangle\right) = \left(|01\rangle - |10\rangle\right).$$

B. The AKLT Hamiltonian:



The 2-body reduced state of the AKLT chain <u>cannot</u> <u>have spin 2</u>!

⇒ Construct 2-body Hamiltonian $h_{i,i+1} = \mathrm{Proj}(S=2)_{i,i+1}$.

We have (w/ $|\psi\rangle$ the AKLT state, $H = \sum_{i=1}^{N} h_{i,i+1}$):

$$h_{i,i+1}|\psi\rangle = 0 \,, \qquad h_{i,i+1} \geq 0$$

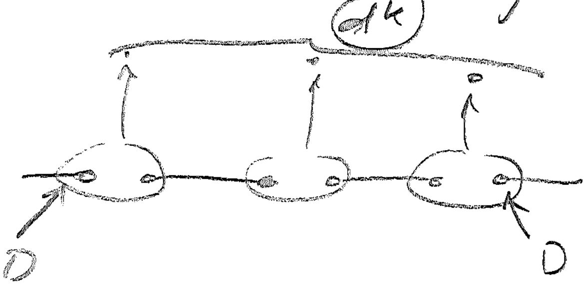
$$\Rightarrow \quad H|\psi\rangle = 0 \;; \qquad H \geq 0$$

⇒ $|\psi\rangle$ is ground state of H.

<u>Note</u>: $h_{i,i+1} \hat{=} \frac{1}{2}\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{6}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{1}{3}$, and thus close to the Heisenberg model (→ cf. exercise)

We will see that $|\psi\rangle$ is the unique G.S. of H, and that H has a gap (⟺ Haldane conjecture!)

<u>Note</u>: We can construct such "parent Hamiltonians" for <u>any</u> MPS: After blocking k sites, the rank of the red. state is always D^2, while the phys. space is d^k-dim. ⇒ if $D^2 > d^k$, RDM has not full rank ⇒ there is always a non-trivial parent Ham.



$$\mathbb{1} - \mathrm{Proj}(\mathrm{range}(\rho_k))$$