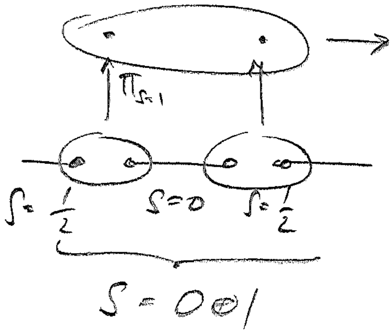
Wrap-up last lecture:

parent Hamiltonian for the AKLT model:



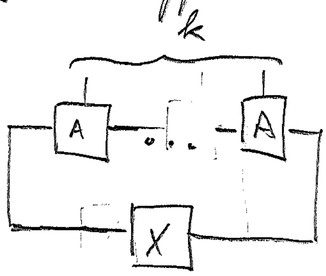
$\rightarrow S = 0 \oplus 1$ (but not $S=2$!)

$$\Rightarrow h_{i,i+1} = \mathrm{Proj}(S=2)_{i,i+1}$$

$$H = \sum h_{i,i+1}$$

$H \geq 0$; $H|\psi_{AKLT}\rangle = 0$; AKLT g. state of H!

For arbitrary MPS: choose k s.th. $d^k > D^2$. The red. DM of k sites is supported on



$$S_k := \left\{ \ \cdots \ \middle| \ X \in M_{D\times D} \right\} = \left\{ \sum_{i_1 \ldots i_k} \mathrm{tr}[A^{i_1} \ldots A^{i_k} X] \, |i_1, \ldots, i_k\rangle \ \middle| \ X \right\}$$

$S_k \subset \mathbb{C}^{(d^k)}$, but rank $S_k \leq D^2 \lneq d^k$

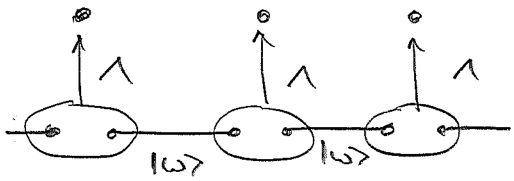
$$\Rightarrow h_{i,\ldots,i+k-1} = \mathbb{1} - \mathrm{Proj}(S_k)_{i,\ldots,i+k-1} \quad \text{non-trivial,}$$

$h_{i,\ldots,i+k-1} |\psi\rangle = 0$ for MPS $\Rightarrow$ ground state.

The fact that $|\psi\rangle$ is ground state of <u>all</u> terms individually is called "<u>frustration free</u>".

C. Uniqueness of ground state

i) Consider state of the form

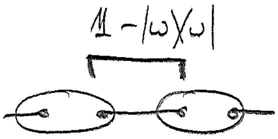
$|\psi\rangle =$ 

with Λ bijective (= invertible): (Relation to AKLT later!)

$$\Rightarrow \quad |\psi\rangle = \Lambda^{\otimes N} \, |\Omega\rangle \; ; \quad |\Omega\rangle = |\omega\rangle^{\otimes N}$$

$|\Omega\rangle$ is unique g.s. of $H = \sum h_{i,i+1}$,

$$h_{i,i+1} = \mathbb{1} \otimes (\mathbb{1} - |\omega\rangle\langle\omega|) \otimes \mathbb{1}$$



Now consider $\tilde{h}_{i,i+1} = (\Lambda_i^{-1} \otimes \Lambda_{i+1}^{-1})^{\dagger} \, h_{i,i+1} \, (\Lambda_i^{-1} \otimes \Lambda_{i+1}^{-1})$.

We have $\tilde{h}_{i,i+1} \geq 0$, and

$$\tilde{h}_{i,i+1} |\psi\rangle = (\Lambda_i^{-1} \otimes \Lambda_{i+1}^{-1})^{\dagger} h_{i,i+1} (\Lambda_i^{-1} \otimes \Lambda_{i+1}^{-1}) (\Lambda \otimes \dots \otimes \Lambda_i \otimes \Lambda_{i+1} \otimes \dots \otimes \Lambda) |\Omega\rangle$$

$$= (\Lambda \otimes \dots \otimes \Lambda \otimes \Lambda_i^{-1} \otimes \Lambda_{i+1}^{-1} \otimes \Lambda \otimes \dots \otimes \Lambda) \underbrace{h_{i,i+1} |\Omega\rangle}_{0} = 0$$

$\Rightarrow |\psi\rangle$ is G.S. of $\tilde{H} = \sum \tilde{h}_{i,i+1}$.

Now consider $|\phi\rangle$ s.th. $\tilde{H}|\phi\rangle = 0$

$\Rightarrow \forall i \quad \tilde{h}_{i,i+1}|\phi\rangle = 0$

$\Rightarrow \forall i \quad (\Lambda_i^{-1}\otimes\Lambda_{i+1}^{-1})^\dagger\, h_{i,i+1}\,(\Lambda_i^{-1}\otimes\Lambda_{i+1}^{-1})|\phi\rangle = 0$

(N.B.: here we use that Λ^{-1} is injective!)

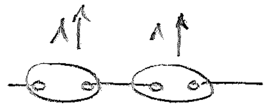
$\Rightarrow \forall i \quad h_{i,i+1}(\Lambda_i^{-1}\otimes\Lambda_{i+1}^{-1})|\phi\rangle = 0$

$\Rightarrow \forall i \quad h_{i,i+1}\left(\bigotimes_k \Lambda_k^{-1}\right)|\phi\rangle = 0$

$\xRightarrow{\text{G.S. of } H=\sum h \text{ unique}} \quad \left(\bigotimes_k \Lambda_k^{-1}\right)|\phi\rangle = |\Omega\rangle$

$\Longrightarrow \quad |\phi\rangle = \left(\bigotimes_k \Lambda_k\right)|\Omega\rangle = |\psi\rangle$

$\Rightarrow$ $|\psi\rangle$ is the only ground state of $\tilde{H}$!

<u>We have seen:</u> If $|\psi\rangle =$ 

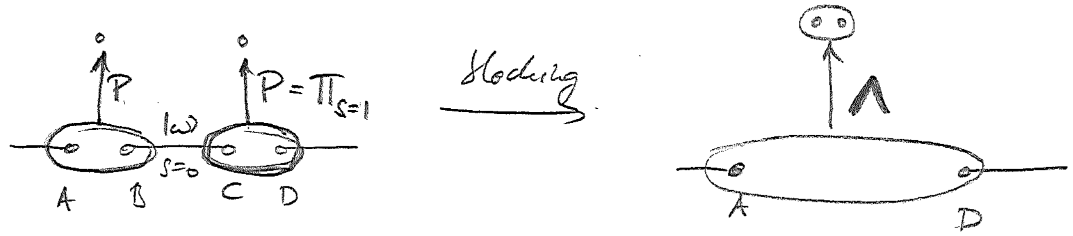
w/ Λ invertible $\Longrightarrow$

$\Rightarrow |\psi\rangle$ unique G.S. of $\tilde{h}_{i,i+1} = (\Lambda_i^{-1}\otimes\Lambda_{i+1}^{-1})^\dagger\, h_{i,i+1}\,(\Lambda_i^{-1}\otimes\Lambda_{i+1}^{-1})$.

<u>Note:</u> 1) All what matters is $\ker \tilde{h}_{i,i+1}$ (since the ground space is $\bigcap_i \ker \tilde{h}_{i,i+1}$), so we can replace $\tilde{h}_{i,i+1}$ by any positive operator w/ identical kernel.

2) If we choose instead a projector, this is equal to the conventional parent Ham. construction!

Relation to the AKLT model:



$$\Lambda := P_{AB} \otimes P_{CD} \, |\omega_{BC}\rangle$$

Claim: Λ is injective (i.e. has a left-inverse).

(Proof: Λ preserves S and S_z $\rightarrow$ the four virtual states on A,D can be identified by their S and S_z quantum numbers.)

$\Rightarrow$ If we restrict each site to range(Λ), Λ is bijective.

$\Rightarrow$ Our proof for unique G.S. applies.

Restricting to range(Λ) is achieved by adding a Ham. term $\mathbb{1} - \mathrm{Proj}(\mathrm{range}(\Lambda))$ per site! Note: This term is automatically obtained from the parent Ham. construction!

$\Rightarrow$ AKLT unique G.S. of local Hamiltonian.

Note: Ham. is 4-body $\Rightarrow$ need to check that 2-body Ham. has same ground space, i.e.

$$\ker h_{12} \cap \ker h_{23} \cap \ker h_{34} = \ker h_{4\text{-body}} \; !$$

($\rightarrow$ Homework!)

D. Gap of the Hamiltonian

H has spectral gap betw. 0 and $\Delta \iff H^2 \geq \Delta H$.

Now consider $H = \sum h_i$, $h_i^2 = h_i$ (projector), h_i nearest neighbor (acting on sites $i, i+1$), and let

$$\boxed{h_i h_{i+1} + h_{i+1} h_i \geq -c\,(h_i + h_{i+1})} \quad (*)$$

Then, H has a gap between 0 and $\gamma = (1-2c)$.

Proof: $H^2 = \left(\sum h_i\right)^2 = \sum \underbrace{h_i^2}_{=h_i} + \underbrace{\sum_i (h_i h_{i+1} + h_{i+1} h_i)}_{\geq -c(h_i + h_{i+1})} + \underbrace{\sum_{|i-j|>1} h_i h_j}_{\geq 0}$

$$\geq \sum h_i - c \sum (h_i + h_{i+1})$$

$$= (1-2c) \sum h_i$$

∎

(Note: Works for any dimension etc. w/ different γ.)

For the AKLT model: $(*)$ with $c < 1/2$ holds for the parent Ham. defined on three sites ($\to$ Homework). Since

$$c_1 h_{123} \geq h_{12} + h_{23} \geq c_2 h_{123} \quad \text{(as kernels equal)},$$

this also gives a bound on the gap of the 2-body Ham.

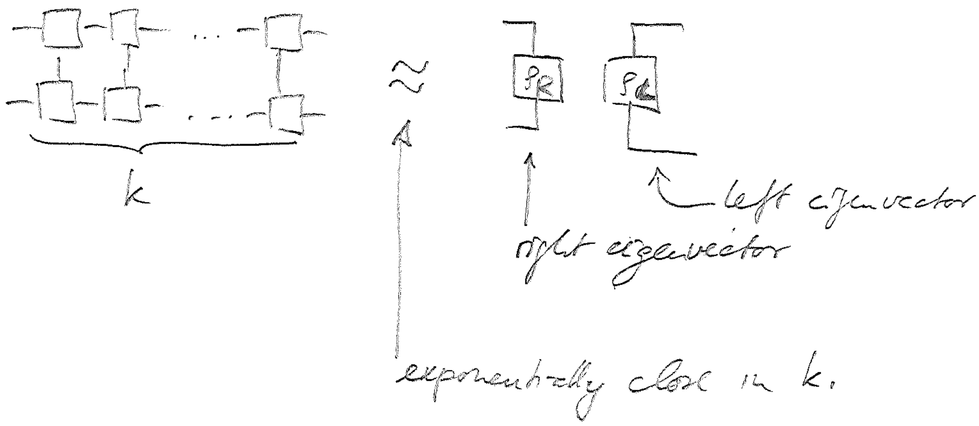
Gap for general MPS

By blocking sites, we can achieve <u>for any MPS</u>

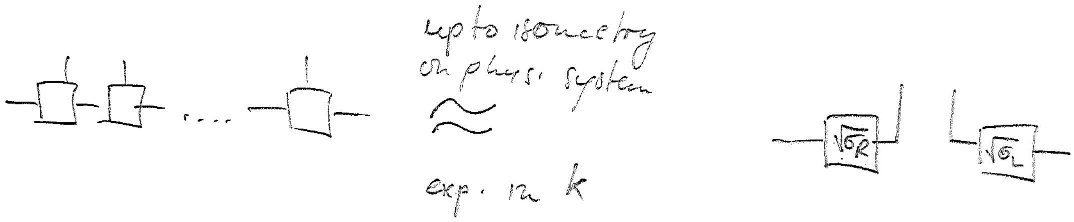
$$h_i h_{i+1} + h_{i+1} h_i \geq -c\,(h_i + h_{i+1})$$

with $c < 1/2$, as long as the MPS is <u>injective</u>, i.e., by blocking tensors we can bring it into a form where  is injective.

Proof sketch:

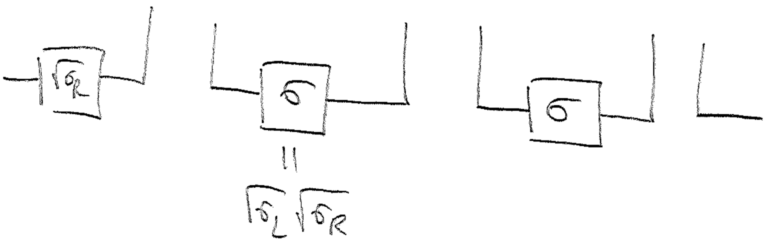
1) Injectivity is equiv. to a unique fixed point of $\mathbb{E} = \sum A^i \otimes \bar{A}^i$, which has full rank (see quant-ph/0608197).
2) By normalization, we can assume $\lambda_{max}(\mathbb{E}) = 1$.
3) 

$\Rightarrow$



4)

For 3 blocked sites: exp. close to



(†)

$\Rightarrow$ parent Ham. of blocked MPS exp. close to parent Ham. of (†).

5) Parent Ham. of (†) commutes, i.e.,

$$h_i h_{i+1} + h_{i+1} h_i \geq 0$$

$\Rightarrow$ parent Ham. satisfies $h_i h_{i+1} + h_{i+1} h_i \geq -c(h_i + h_{i+1})$ for some small c. (careful!)

All arguments assume that all these approx. behave nicely. Vaguely speaking, this is true since all spaces are finite-dim. $\Leftrightarrow$ all norms are equivalent.

<u>Disclaimer:</u> This is only a sketch of the argument — — no guarantees taken!