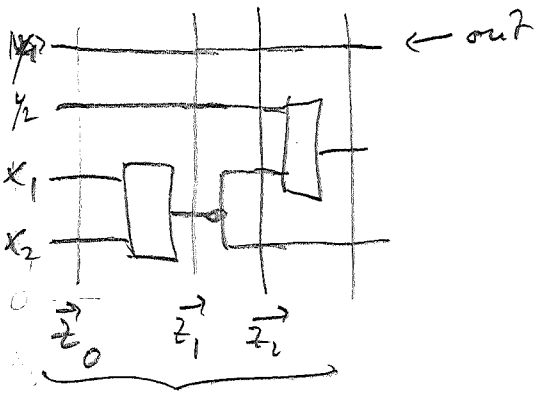
Quantum complexity of many-body systems

Reminder of Lecture 13:

* Complexity theory: classify difficulty of problems
* Complexity class P: Class of problems which can be solved in time poly(n), with n the problem size (= # bits needed to specify it).
* Class NP: Class of problems where solution can be checked in time poly(n): i.e., $L \in NP :\Leftrightarrow$ there exists poly-time computable verifier $g(x,y) \in \{0,1\}$ s.t. $\forall x \in L \; \exists y : g(x,y) = 1$, $\forall x \notin L \; \forall y : g(x,y) = 0$.
* NP-complete problem: Problem which is (at least) as hard as any other problem in NP, i.e., solving any NP problem can be reduced to solving the complete problem.
* If NP ≠ P (commonly believed), then NP-complete problems cannot be solved in time poly(n).

* typ. NP-complete problem: k-SAT: Given variables $x_1, \dots, x_n$ and poly(n) k-body constraints, can they be simultaneously be satisfied?

Idea of proof: history of verifier checking proof



correct values of x_i, of "out" (= accept), and correct action of gates (incl. identity!) can be enforced by 3-bit constraints!

$\Leftrightarrow$ all constraints can be satisfied $\nexists y : g(x,y)=1$

$\Rightarrow$ 3-body class Hamiltonians (on 2D lattice) are NP-complete (i.e., hard!)

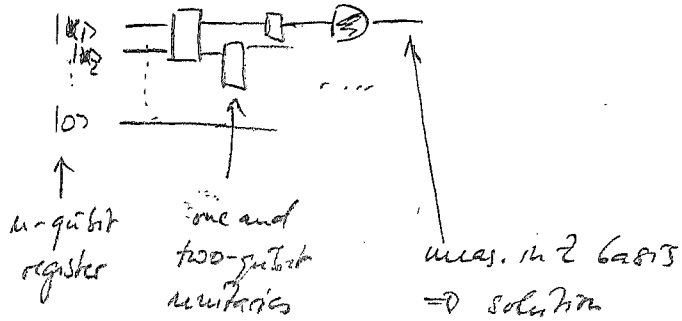
What about quantum Hamiltonians – are they even harder?

Quantum complexity:

BQP ("bounded-error quantum polynomial time"):

class of problems which can be decided by a quantum computer in time poly($|x|$) w/ bounded error (e.g. yes: prob(out=1) $\geq 2/3$, no: — " — $\leq 1/3$)

Reminder – cf. lecture 12: Q. Computer:



- BQP $\leftrightarrow$ problems cf. solvable by q. computer.
- Clearly, $P \subset BQP$.
- Factoring: $\in BQP$, not known to be in P (Shor's algorithm)

QMA ("Quantum Merlin Arthur"):

Q. version of NP: For "yes" instances, there exists a quantum proof $|\psi\rangle$ (of size poly(n)) which can be checked efficiently (time poly(n)) by a quantum computer. (w/ bounded error probability).

Example for QMA problem:

* "LOCAL HAMILTONIAN": given $H = \sum h_i$, h_i few-body terms ("local"): Is the g.s. energy of H below some a ("yes") or above some b ("no"), where $b - a > \frac{1}{\text{poly}(n)}$

(Roughly: Determine g.s. energy of H up to $1/\text{poly}(n)$)

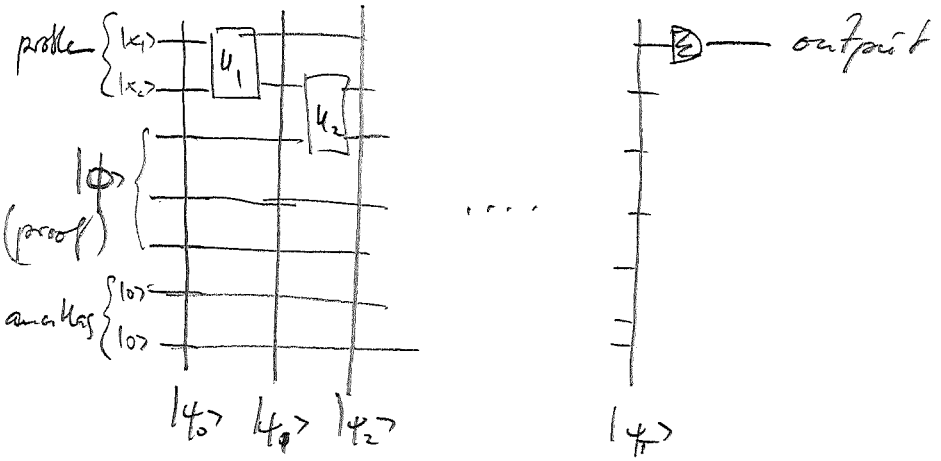
The quantum proof $|\psi\rangle$ can be any state w/ energy $\langle\psi|H|\psi\rangle < a$. (e.g. the g. state) — if it exists. ($\langle\psi|H|\psi\rangle$ can be determined to $\frac{1}{\text{poly}(n)}$ using phase estimation.)

Turns out: Local Ham. is QMA-complete, i.e., among the hardest problems in QMA: General G.S. problems cannot be solved efficiently <u>even</u> by Q.

QMA-completeness of "local Hamiltonian"

Similar to Cook-Levin proof:

General QMA problem $\longleftrightarrow$ verifier circuit



Build again <u>history state</u> $|\psi\rangle = \sum |\psi_t\rangle |t\rangle_t$.

Hamiltonian terms:

$H_{init} = \sum h_i \otimes |0\rangle\langle 0|_t$ → penalize wrong x_i & $|0\rangle$ ancillas.

$H_{final} = \ldots |0\rangle\langle 0|_1 \otimes |T\rangle\langle T|_t$ → penalize "no" output

$H_{prop} = \sum_t -\left(U_t \otimes |t\rangle\langle t-1| + h.c.\right) + |t\rangle\langle t| + |t-1\rangle\langle t-1|$

→ penalize "incorrect propagation" (q.s. has $|\psi_t\rangle = U_t |\psi_{t-1}\rangle$).

$$\Rightarrow \boxed{H_{QMA} = H_{init} + H_{final} + H_{prop}}$$

H_{QMA} has zero-energy G.S. <u>exactly iff</u> there exists a <u>proof</u> $|\phi\rangle$ — otherwise, either output is "no", or propagation is wrong, or ancillas $|\psi_i\rangle$ are strongly entangled.

$\Rightarrow$ energy difference $1/\text{poly}(L)$ betw. yes/no instances

$\Rightarrow$ H_{QMA} QMA-complete.

<u>Notes:</u>

* H_{QMA} is $\log N$-local (i.e. terms in H involve $\log N$-many terms) since time reg. has $\log N$ qubits $\rightarrow$ can be reduced to 5-local ($\rightarrow$ homework)

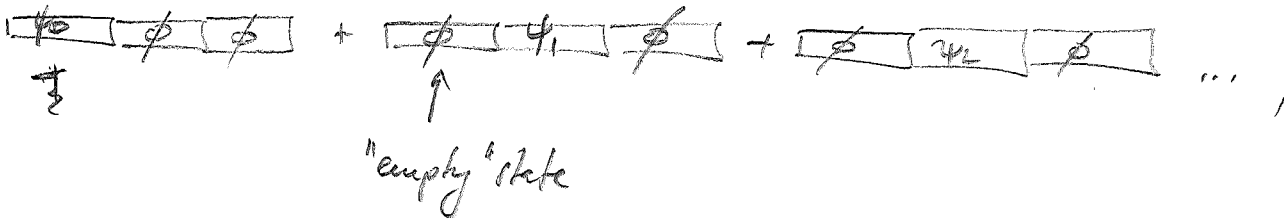
* simpler Hams can be constructed using so-called "perturbation gadgets" ($\rightarrow$ homework)

  $\Rightarrow$ 2D square lattice of qubits w/ 2-~~body~~ Pauli product couplings is still QMA-complete!

* Extremely surprising: 1D NN Ham. is <u>still</u> QMA-cpl.

  ($\longleftrightarrow$ class 1D: P).

(Idea: Instead of time register use space: history state =



)

Why did we not use a history state

$$|\psi_0\rangle \otimes |\psi_1\rangle \otimes \dots \otimes |\psi_T\rangle$$

just as classically?

$\Rightarrow$ impossible to guarantee that $|\psi_t\rangle = U_t |\psi_{t-1}\rangle$,

since we can "cheat" by using entangled states

(and we cannot ensure the proof has product form);

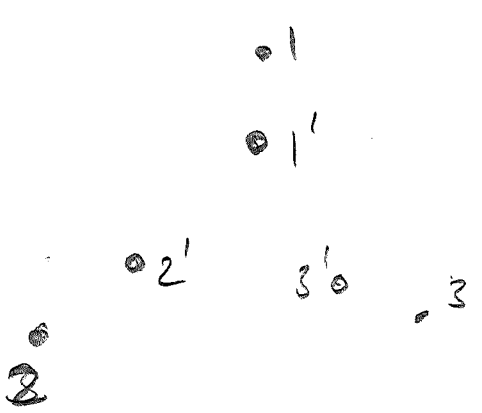
alternatively: That would violate the no-cloning principle.

Sketch: Imagine $U_t = \mathbb{1}$, i.e. $|\psi_t\rangle = |\psi_{t-1}\rangle$. We want to have a state w/ ground space spanned by $|\psi_t\rangle \otimes |\psi\rangle \Rightarrow$ That is the full symmetric subspace. But this also contains $|0\rangle|1\rangle + |1\rangle|0\rangle$

$\Rightarrow$ ↯!

Basic idea of perturbation gadgets:

6 qubits:



We want 3-body interaction on 1+2+3!

$$H = -I_1 I_2 I_3 \underbrace{\left(Z_1 Z_2 I_{3'} + Z_1 I_{2'} Z_{3'} + I_{1'} Z_2 Z_{3'} - 3 I_1 I_2 I_3 \right)}$$

ground space: $|000\rangle, |111\rangle$, $E=0$

gap $\Delta = 2$, all ex. states have same energy,

$$V = \gamma \left(B_1 \otimes X_{1'} + B_2 \otimes X_{2'} + B_3 \otimes X_{3'} \right) \; ; \; \gamma \ll 1.$$

Pert. theory:

1st order: 0

2nd order: $\frac{\gamma^2}{\Delta} \left(B_1^2 + B_2^2 + B_3^2 \right)$

3rd order: $\frac{\gamma^3}{\Delta^2} \left(B_1 B_2 B_3 + B_1 B_3 B_2 + \dots \right) = \underbrace{6 B_1 B_2 B_3}$

3-body term!