Lecture *Entanglement and Locality in Quantum Many-body Systems* — Exercise Sheet #1

Problem 1 ((a)–(c) easy, (d) a bit tricky)

We have defined the operator norm as $\|A\| = \sup_{|\phi\rangle} \frac{|A|\phi\rangle|}{||\phi\rangle|}$. Show that

(a) $|\,A|\phi\rangle\,| \le \|A\|\,|\,|\phi\rangle\,|$.

(b) For U and V unitary, $\|VAU\| = \|A\|$.

(c) $\|AB\| \le \|A\|\,\|B\|$.

(d) Show that $\|A \otimes \mathbb{1}\| = \|A\|$.
(*Hint:* The "$\ge$" direction should be easy. To prove the "$\le$" direction, use that any normalized vector on a bipartite system can be written as $|\phi\rangle = \sum_i \sqrt{p_i}|\alpha_i\rangle|\beta_i\rangle$, with $p_i \ge 0$, $\sum p_i = 1$, and $\langle\alpha_i|\alpha\rangle_j = \langle\beta_i|\beta\rangle_j = \delta_{ij}$.)

Problem 2 (easy to medium)

For any operator A on a lattice Λ, we can construct an operator $\tilde{A}$ which acts only a sub-region Z through

$$\tilde{A} := \int \mathrm{d}U U A U^\dagger \ .$$

Here, $\int \mathrm{d}U$ is an integral over all unitary matrices acting on $\Lambda\backslash Z$, where the integral is over the so-called Haar measure (or unitarily invariant measure), which means that the integral has the property that

$$\int \mathrm{d}f(U) = \int \mathrm{d}f(VU) = \int \mathrm{d}f(UV)$$

for any unitary V acting on $\Lambda\backslash Z$.

(a) Show that the unitary invariance of the integral implies that $V\tilde{A} = \tilde{A}V$ for any unitary V acting on $\Lambda\backslash Z$.
It turns out that this property implies that $\tilde{A}$ is of the form $\tilde{A}_Z \otimes \mathbb{1}_{\Lambda\backslash Z}$, i.e., $\tilde{A}$ is an operator supported on Z. This is a consequence of Schur's lemma (a fundamental lemma in representation theory), and we are not going to prove this here. Rather, we want to show that $\tilde{A}$ provides a good approximation to A, given a Lieb-Robinson type bound on commutators.

(b) Consider operators $A_X(t)$ and B_Y as they appear in the Lieb-Robinson bound, let $K_l(X)$ be the circle of radius l around X, and define $\tilde{A}_X(t)$ as above with U supported in $\Gamma\backslash K_l(X)$. Show that

$$\|\tilde{A}_X(t) - A_X(t)\| \le \int \mathrm{d}U \|[A_X(t), U]\|$$

and argue how this shows that $A_X(t)$ is well approximated by an operator $\tilde{A}_X(t)$ which is supported in $K_l(X)$ given a Lieb-Robinson bound holds.
(*Hint:* Use that $UAU^\dagger = A + U[A, U^\dagger]$.)

Problem 3 (medium)

Show that the Lieb-Robinson bound given in the Lecture gives rise to the following bound (under the same conditions as the original Lieb-Robinson bound): There exists a velocity v such that for all $t \le l/v$, it holds that

$$\|[A_X(t), B_Y]\| \le \frac{vt}{l} g(l) |X|\, \|A_X\|\, \|B_Y\| \ ,$$

where $l = d(X, Y)$, and $g(l)$ decays exponentially with l. This is, outside the "light cone" $l = vt$, the correlations decay exponentially with the distance (and linearly with the "angle" vt/l).
(*Hint:* Choose a velocity $v = \alpha(2s/\mu)$ with $\alpha > 1$, and use $e^x - 1 \le xe^x$.)
How does the rate of the exponential decay of $g(l)$ depend on the chosen velocity v, i.e., on the α above?