Lecture *Entanglement and Locality in Quantum Many-body Systems* — Exercise Sheet #3

Problem 1: Fourier transforms

a) Consider a function $f(t)$ which is k times differentiable, with $f^{(\ell)}(t) \to 0$ for $|t| \to \infty$, and $|f^{(k)}(t)|$ integrable. Show that $\hat{f}(\omega)$ decays as ω^{-k}.
Show that conversely, if $f(t)$ decays as $t^{k+\epsilon}$, f is $k-1$ times continuously differentiable.

b) In *A. Ingham, J. London Math. Soc.* ***9****, 29 (1934)* (available at `http://dx.doi.org/10.1112/jlms/s1-9.1.29`), it is shown that one can construct a function $g(t)$ with $\hat{g}(\omega) = 0$ for $|\omega| \geq 1$ such that $g(t)$ decays as $e^{-t\,\epsilon(t)}$ as long as $\int_c^\infty \epsilon(t)/t\,\mathrm{d}t$ does not diverge at infinity. Show that this holds for $\epsilon(t) = 1/(\log t)^2$, but not for $\epsilon(t) = 1/\log t$.

c) Show that given a function $g(t)$ as above, the function $f(t) := \delta(t) - g(t)$ fulfils $\hat{f}(\omega) = 1$ for $|\omega| \geq 1$ and has the same decay as $g(t)$ for large t. Show that we can always find an even function f with the same properties (even if g is not even). Also check that if $\hat{g}(0) = 1$, then $\hat{f}(0) = 0$.

d) Given any even function $f(t)$ with $\hat{f}(0) = 0$, show that

$$F(t) = \frac{1}{2i} \int \mathrm{d}u\, f(u)\, \mathrm{sgn}(t-u)$$

with $\mathrm{sgn}(t) = t/|t|$, $\mathrm{sgn}(0) = 0$ the sign function, satisfies

i) $|F(t)| \leq \left| \int_{|t|}^{\infty} f(u)\,\mathrm{d}u \right|$, i.e., $F(t)$ decays rapidly if $f(t)$ decays rapidly.

ii) $\hat{F}(\omega) = \frac{1}{\omega} \hat{f}(\omega)$, i.e., for $f(t)$ as above, $F(t)$ behaves as required in the lecture. (This can be proven either by interpreting the integral in Fourier space, or by integration by parts.)

Problem 2: Quasi-adiabatic evolution of degenerate ground spaces

a) Consider a Hamiltonian $H(s)$ with a set of non-degenerate ground states $|\psi_0^\alpha(s)\rangle$ with energies $E_0^\alpha(s)$ ($E_0^\alpha(s) \neq E_0^\beta(s)$ for $\alpha \neq \beta$), separated from the other eigenvalues by a gap Δ, i.e., $E_i(s) - E_0^\alpha(s) \geq \Delta$. Show that the time evolution of the $|\psi_0^\alpha(s)\rangle$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}s} |\psi_0^\alpha(s)\rangle = i\mathcal{D}_s |\psi_0^\alpha(s)\rangle + \sum_{\beta \neq \alpha} Q_{\alpha\beta} |\psi_0^\beta(s)\rangle \ ,$$

with $\mathcal{D}_s$ the quasi-adiabatic evolution operator defined in the lecture,

$$i\mathcal{D}_s = \int F(\Delta\, t) e^{iH(s)t} \left(\frac{\mathrm{d}}{\mathrm{d}s} H(s) \right) e^{-iH(s)t}\, \mathrm{d}t \ .$$

Determine $Q_{\alpha\beta}$ and verify it is anti-hermitian.
Consider what happens if the low-lying eigenstates $|\psi_0^\alpha(s)\rangle$ are exactly degenerate. Can this be resolved by choosing an appropriate basis of eigenstates?

b) Use the result of a) to derive an evolution equation for the projector onto the ground space,

$$P_0(s) = \sum_\alpha |\psi_0^\alpha(s)\rangle\langle\psi_0^\alpha(s)| \ .$$

What happens in this case if the eigenstates become degenerate?