**Lecture *Entanglement and Locality in Quantum Many-body Systems* — Exercise Sheet #5**

## Problem 1

Generalize the proof of the Singular Value Decomposition given in the lecture to the case of non-square matrices, and matrices $M$ where both $MM^\dagger$ and $M^\dagger M$ do not have full rank.

## Problem 2

Find the Schmidt decompositions and the reduced density matrices for the following states on two qubits. You can try to find the Schmidt decomposition either using the singular value decomposition or the reduced density matrices.

$$\frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} \tag{1}$$

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \tag{2}$$

$$\frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2} \tag{3}$$

$$\frac{|00\rangle + 2|01\rangle + 2|10\rangle - \alpha|11\rangle}{\sqrt{9+\alpha^2}} \tag{4}$$

It is completely fine to use a computer algebra system, in particular for the last one.

## Problem 3

Let $\{p_i\}$ be a probability distribution, i.e., $\sum p_i = 1$.

a) Check that the Rényi entropy

$$S_\alpha(\{p_i\}) = \frac{\log(\sum p_i^\alpha)}{1-\alpha}$$

converges to the von Neumann entropy

$$S(\{p_i\}) = -\sum p_i \log p_i$$

for $\alpha \to 1$.

b) Show that the Rényi entropy is monotonically decreasing in $\alpha$, i.e.,

$$\alpha \geq \beta \quad \Rightarrow \quad S_\alpha(\{p_i\}) \leq S_\beta(\{p_i\}) \ \forall \{p_i\} \quad .$$

Does strict monotonicity hold?
(*Hint:* First, show that $\frac{\mathrm{d}}{\mathrm{d}\alpha} S_\alpha(\{p_i\})$ is proportional to the Kullback-Leibler divergence $D(\{q_i\}||\{p_i\}) = \sum q_i \log(q_i/p_i)$, where $q_i = p_i^\alpha / \sum p_i^\alpha$. Second, show that $D(\{q_i\}||\{p_i\}) \geq 0$, using that $\log x \geq 1 - 1/x$).

## Problem 4

Use the iterative construction introduced at the end of Lecture 10 to find a Matrix Product State representation for

a) the GHZ state $|0\dots0\rangle + |1\dots1\rangle$

b) the W state $|100\dots0\rangle + |010\dots0\rangle + |001\dots0\rangle + \cdots + |000\dots1\rangle$

for a chain of length $N$. (It might be a good idea to start by considering short chains, and then generalize the result.)