Quasi-adiabatic evolution

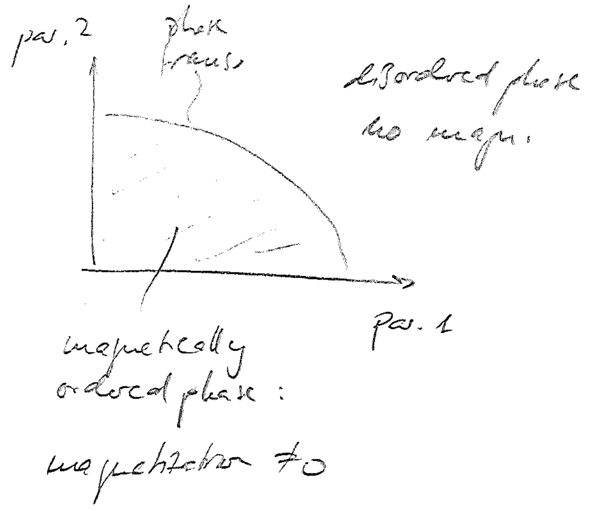
Phases: Qualitatively different states of matter, with discontinuous phase transitions betw. them (as a function of temp. & ext. parameters)

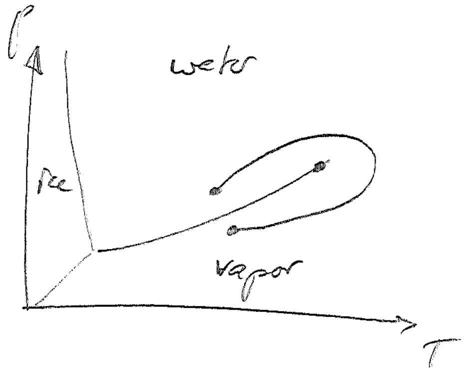
Quantum phases / phase transitions (QPT): at zero temperature (ground state)

Characterization of phases:

- by some property ("order parameter")
  symmetry breaking or long-range order,
  crystal structure, magnetic ordering, superconductivity, ...

- by equivalence classes: two points in parameter space are in the same phase if we can go continuously from $A \to B$ w/out a phase transition (i.e., w/out "discontinuous behavior").

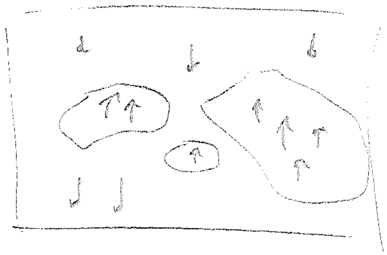




Note: Can give different classifications. Equiv. classes might also distinguish phases w/o order parameters (→ next lecture!)

Phase transitions:

- Different types
- 2nd order; e.g. 2D class. Ising model at $T=T_c$:



clusters at all sizes form. → absence of length scale

→ diverging correlation length

$\xrightarrow{QPT}$ gap must close.

Common definition of QPT: Gap in H closes.

i.e.: $H^0 = \sum_z h_z^0$ and $H^1 = \sum_z h_z^1$ are in the same phase

$\iff \exists$ smooth path $H(\theta) = \sum_z h_z(\theta)$ of local Hams

s.th. $H(0) = H^0$, $H(1) = H^1$, and $H(\theta)$ is gapped (i.e.,

$\exists \Delta$ s.th. $\mathrm{Gap}(H(\theta)) \geq \Delta \quad \forall N \, \forall \theta$.)

What does this imply about the structure of the ground states of H^0 and H'? Is there <u>any relation</u>?

<u>This lecture:</u> Gapped path $\Rightarrow$ G.S. related by quasi-local transformation $\Rightarrow$ no change in global structure (e.g. long-range correlations, global entanglement structure, ...).

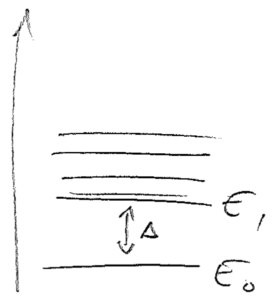
Relation via <u>time evolution</u> w/ quasi-local "Hamiltonian".
("Quasi-adiabatic evolution": cf. adiabatic evolution: "inf. slow" evolution with $H(\theta)$ which preserves the ground state.)

<u>Quasi-adiabatic evolution:</u>

Consider $H' = H + \varepsilon V$.

Eigenstates $|\psi_i(\varepsilon)\rangle$, Energies $E_i(\varepsilon)$.

<u>Gap:</u> $E_i - E_0 \geq \Delta; \quad i \neq 0$



(<u>Note</u>: Generalizes to > 1 ground states.)

Define

$$i\mathcal{D}(H,V) = \int_{\mathbb{R}} F(\Delta t)\, e^{iHt} V e^{-iHt}\, dt$$

We want $F(t)$ with:

- $F(-t) = -F(t)$ $\quad (\Rightarrow \hat{F}(0) = 0)$
- $\hat{F}(\omega) = -\frac{1}{\omega}$ for $|\omega| \geq 1$.
- $\hat{F}(\omega)$ is very smooth ($\mathcal{C}^\infty$, e.g.):

  $|F(t)|$ decays faster than any polynomial for $|t| \to \infty$.

  (Useful fact: $\hat{F} \in \mathcal{C}^k \Leftrightarrow F$ decays faster than $\frac{1}{t^{k+\dim}}$)

  (In fact, we can obtain $|F(t)| \sim e^{-t/(\log t)^2}$.

Such functions exist — cf. HW sheet 3.

(N.B.: Other choices also work, e.g. based on Gaussians.)

What is $\frac{d}{d\varepsilon} |\psi_0(\varepsilon)\rangle \big|_{\varepsilon=0}$?

$$\underline{\frac{d}{d\varepsilon} |\psi_0(\varepsilon)\rangle \Big|_{\varepsilon=0}} = \sum_{i \neq 0} \underbrace{\frac{1}{E_i(0) - E_0(0)}} |\psi_i(0)\rangle\langle\psi_i(0)|V|\psi_0(0)\rangle$$

$$= \frac{1}{\Delta} \hat{F}\left(\frac{E_i(0) - E_0(0)}{\Delta} \right)$$

$$= \int F(\Delta t)\, e^{i(E_i(0) - E_0(0))t} \, dt$$

$$= \sum_{i\neq 0} |\psi_i(0)\rangle\langle\psi_i(0)|V|\psi_0(0)\rangle \underbrace{\int F(\Delta t)\, e^{i(E_i(0)-E_0(0))t}\, dt}_{=0 \text{ for } i=0 \Rightarrow \text{ add to } \sum_{i\neq 0}}.$$

$$= \underbrace{\sum |\psi_i(0)\rangle\langle\psi_i(0)|}_{=\mathbb{1}} \int F(\Delta t)\, e^{iH(0)t}\, V e^{-iH(0)t}\, dt\, |\psi_0(0)\rangle$$

$$= \underline{i\mathcal{D}(H,V)\,|\psi_0(0)\rangle}.$$

More generally: Consider parameter-dependent Hamiltonian

$H(s)$, with g.s. $|\psi_0(s)\rangle$ + gap.

Define $\mathcal{D}_s = \mathcal{D}\left(H(s), \frac{d}{ds}H(s)\right) = \frac{1}{i}\int F(\Delta t)\, e^{iH(s)t} \frac{d}{ds}H(s)\, e^{-iH(s)t}\, dt$

Then: $H(s+\varepsilon) \approx H(s) + \left(\frac{d}{ds}H(s)\right)\varepsilon + \dots$

$$\Rightarrow \quad \boxed{i\mathcal{D}_s|\psi_0(s)\rangle = \frac{d}{ds}|\psi_0(s)\rangle}$$

Time evolution with the quasi-adiabatic evolution operator transforms $|\psi_0(s=0)\rangle$ to $|\psi_0(s=1)\rangle$.

Similar version for degenerate/non-degen. ground spaces exist! (→ Homework)

Non-degen. G.S.:

$$\frac{d}{ds}|\psi_0^\alpha(s)\rangle = i\mathcal{D}_s|\psi_0^\alpha(s)\rangle + \sum' Q_{\alpha\beta}|\psi_0^\beta(s)\rangle$$

Degen. G.S.:

$$\frac{d}{ds}P_0(s) = i\left[\mathcal{D}_s, P_0(s)\right]; \quad P_0(s) = \sum_\alpha |\psi_0^\alpha(s)\rangle\langle\psi_0^\alpha(s)|$$

Essential question: Does $\mathcal{D}_s$ preserve locality? Is it a local Hamiltonian, or does it otherwise obey a LR-type bnd.?

Will show: Yes.

To this end: $H(s) = \sum_Z h_Z(s)$, with $\|h_Z(s)\|$ rapidly decaying in diam (Z) (s.th. LR-bnd holds), and bounded $\frac{d}{ds}h_Z(s)$ (fixes time scale!) – e.g. $\|\frac{d}{ds}h_Z(s)\| \leq c\cdot\|h_Z(s)\|$.

Define $$\mathcal{D}_s(Z) := \int dt\, F(\Delta t)\, \underbrace{e^{iH(s)t}\overbrace{\frac{d}{ds}h_Z(s)}^{=:Q}e^{-iH(s)t}}_{=:Q(t)}$$

$$\Rightarrow \mathcal{D}_s = \sum_Z \mathcal{D}_s(Z).$$

But: $\mathcal{D}_s(Z)$ is not supported on Z!

Idea: Decompose $\mathcal{D}_S(z)$ into local terms:

For small t: LR-bound on $Q(t)$

For large t: $|F(\Delta t)| \to 0$ super-polynomially!

Concretely: Exercise Sheet 4, Problem 2:

$$Q^{\ell}(t) = \mathrm{tr}_{\Lambda \setminus B_\ell(z)}\, Q(t) \otimes \mathbb{1}_{\Lambda \setminus B_\ell(z)}$$

$\to$ support on ball $B_\ell(z)$ of radius ℓ around z.

$$\to \| Q^{\ell}(t) - Q(t) \| \leq c \cdot |X| \cdot \|Q\| \, e^{-\mu \ell}, \quad vt < \ell.$$

$$\leq 2\|Q\|, \quad vt > \ell.$$

also works for slower decay!

Write: $Q(t) = (Q^0(t) - Q^1(t)) + (Q^1(t) - Q^2(t)) + (Q^2(t) - Q^3(t)) + \ldots$

$$\Rightarrow \mathcal{D}_S(z) = \underbrace{\int F(\Delta t)(Q^0(t) - Q^1(t))\, dt}_{S^1} + \underbrace{\int F(\Delta t)(Q^1(t) - Q^2(t))\, dt}_{S^2} + \ldots$$

(1) S^ℓ supported on $B_\ell(z)$!

(2) $\|S^\ell\|$ decays rapidly with ℓ (if $\ell > vt$):

$$\|Q^{\ell-1}(t) - Q^\ell(t)\| \leq \|Q^{\ell-1}(t) - Q(t)\| + \|Q(t) - Q^\ell(t)\|$$

$$\leq \begin{cases} 2c|X|\|Q\| \, e^{-\mu\ell} & , vt < \ell \\ 4\|Q\| & , vt > \ell . \end{cases}$$

$$\Rightarrow \|S^\ell\| \leq \int |F(\Delta t)| \, \|Q^{\ell-1}(t) - Q^\ell(t)\| \, dt$$

$$\leq \underbrace{\int_{|t| < \ell/v} |F(\Delta t)| \cdot 2c|X|\|Q\| \, e^{-\mu\ell} \, dt}_{\leq c|X|\|Q\| \ell \, e^{-\mu\ell}} \quad +$$

$$+ \underbrace{\int_{|t| > \ell/v} \underbrace{|F(\Delta t)|}_{\leq \frac{1}{t^\alpha} \, \forall \alpha} \cdot 4\|Q\| \, dt}_{\leq \frac{c}{t^\alpha} \|Q\|} .$$

... and $\|Q\| \leq c \|h_Z(s)\|$.

$\Rightarrow$ decays superpolynomially in ℓ, i.e., claim (2).

$$\Rightarrow \mathcal{D}_s = \sum_Z D_{s,Z} \quad \text{(collect terms!)}$$

where ① $D_{s,Z}$ supp. on Z.

② $D_{s,Z}$ decays super-poly. w/ diam(Z).

<u>Important consequence</u>:

$\mathcal{D}_s$ itself obeys a LR-bnd (but <u>no</u> v!!),

$\Rightarrow$ quasi-adiabatic evolution maps local operators to local operators (since path $\hat{=}$ time s is finite!)

$\Rightarrow$ no global changes in state along gapped path!

... more next lecture.