Exercise Sheet # 1

Problem # 01 (Pauli Matrices).

① $X = X^\dagger$, $Y = Y^\dagger$, $Z = Z^\dagger$ (Hermitian).

$XX^\dagger = X^\dagger X = YY^\dagger = Y^\dagger Y = ZZ^\dagger = Z^\dagger Z = I$ (unitary).

$X^2 = Y^2 = Z^2 = I$.

$\{X, Y\} = XY + YX = \begin{pmatrix} i & \\ & -i \end{pmatrix} + \begin{pmatrix} -i & \\ & i \end{pmatrix} = 0$

$\{X, Y\} = \{Y, Z\} = \{Z, X\} = 0$

---

② $i = j \Rightarrow tr(\sigma_i \sigma_j) = tr(\sigma_i^2) = tr(I) = 2$.

$i \neq j \Rightarrow tr(\sigma_i \sigma_j) = tr(\sigma_k) = 0$

$\hookrightarrow k \neq 0$

$\Rightarrow tr(\sigma_i \sigma_j) = 2\delta_{ij}$

---

③ $X = |0\rangle\langle 1| + |1\rangle\langle 0|$.

$Y = i|0\rangle\langle 1| - i|1\rangle\langle 0|$

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$.

---

④ Z, $|0\rangle, |1\rangle$ (Eigenvectors).

$\sigma(Z) = \{1, -1\}$ (Eigenvalues).

X, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$\sigma(X) = \{1, -1\}$.

Y, $|+y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$.

$\sigma(Y) = \{1, -1\}$

(5) $P_1 = |+y\rangle\langle +y|$

$P_{-1} = |-y\rangle\langle -y|$

$$p_1 = \langle\psi|P_1|\psi\rangle$$

Probability of outcome 1

$$= \langle\psi|+y\rangle\langle +y|\psi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\alpha^*\langle 0| + \beta^*\langle 1|\right)\left(|0\rangle + i|1\rangle\right)\ (h.c)$$

$$= \frac{1}{\sqrt{2}}\left(\alpha^* + i\beta^*\right)\frac{1}{\sqrt{2}}\left(\alpha - i\beta\right)$$

$$= \tfrac{1}{2}\left(1 - i\alpha\beta^* + i\alpha^*\beta\right)$$

$$p_{-1} = \langle\psi|P_{-1}|\psi\rangle$$

$$= \langle\psi|-y\rangle\langle -y|\psi\rangle$$

$$= \tfrac{1}{2}\left(1 + i\alpha\beta^* - i\alpha^*\beta\right)$$

$$|\psi\rangle \xrightarrow{\text{outcome "1"}} \frac{P_1|\psi\rangle}{\sqrt{p_1}} = \frac{\langle +y|\psi\rangle\,|+y\rangle}{\sqrt{p_1}}$$

$$|\psi\rangle \xrightarrow{\text{"-1"}} \frac{P_{-1}|\psi\rangle}{\sqrt{p_{-1}}} = \frac{\langle -y|\psi\rangle\,|-y\rangle}{\sqrt{p_{-1}}}$$

---

(6)

$$X\otimes X = \begin{pmatrix} 0\begin{pmatrix}0&1\\1&0\end{pmatrix} & 1\begin{pmatrix}0&1\\1&0\end{pmatrix} \\ 1\begin{pmatrix}0&1\\1&0\end{pmatrix} & 0\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

$$Z\otimes X = \begin{pmatrix} 1\begin{pmatrix}0&1\\1&0\end{pmatrix} & 0\begin{pmatrix}0&1\\1&0\end{pmatrix} \\ 0\begin{pmatrix}0&1\\1&0\end{pmatrix} & -1\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}$$

Problem # 02 (Bloch Sphere for pure States).

① $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$= e^{i\alpha_1}\gamma_1|0\rangle + e^{i\alpha_2}\gamma_2|1\rangle$

Since $|\alpha|^2 + |\beta|^2 = 1$

$\Rightarrow \gamma_1^2 + \gamma_2^2 = 1$

Also, $\cos^2\theta/2 + \sin^2\theta/2 = 1$

$\Rightarrow |\psi\rangle = e^{i\alpha_1}\cos\theta/2\,|0\rangle + e^{i\alpha_2}\sin\theta/2\,|1\rangle$

$= e^{i\alpha_1}\left(\cos\theta/2\,|0\rangle + e^{i(\alpha_2-\alpha_1)}\sin\theta/2\,|1\rangle\right)$

$= e^{i\alpha_1}\left(\cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle\right)$

implies convenient reps. of important states

---

② X, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \Rightarrow (\theta,\varphi) = (\pi/2, 0)$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \Rightarrow$ " $= (\pi/2, \pi)$

Y, $|+y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ $(\theta,\varphi) = (\pi/2, \pi/2)$

$|-y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ " $= (\pi/2, 3\pi/2)$.

Z, $|0\rangle$ $(\theta,\varphi) = (0, [0,2\pi])$

$|1\rangle$ " $= (\pi, [0, 2\pi])$

③ $|\psi\rangle\langle\psi| = \left(\cos(\theta/2)|0\rangle + e^{i\varphi}\sin\theta/2\,|1\rangle\right)\left(\cos\theta/2\,\langle 0| + e^{-i\varphi}\sin\theta/2\,\langle 1|\right)$

$= \cos^2\theta/2\,|0\rangle\langle 0| + e^{-i\varphi}\sin\theta/2\cos\theta/2\,|0\rangle\langle 1| +$

$e^{i\varphi}\sin\theta/2\cos\theta/2\,|1\rangle\langle 0| + \sin^2\theta/2\,|1\rangle\langle 1|.$

$$= \begin{pmatrix} \cos^2\theta/2 & e^{-i\varphi}\sin\theta/2\cos\theta/2 \\ e^{i\varphi}\sin\theta/2\cos\theta/2 & \sin^2\theta/2 \end{pmatrix}$$

$$= \begin{pmatrix} (1+\cos\theta)/2 & (\cos\varphi - i\sin\varphi)\frac{\sin\theta}{2} \\ (\cos\varphi + i\sin\varphi)\frac{\sin\theta}{2} & (1-\sin\theta)/2 \end{pmatrix}$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \cos\theta\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} + \cos\varphi\sin\theta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\varphi\sin\theta\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]$$

$$= \frac{1}{2}\left(\mathbb{1} + (\cos\theta, \cos\varphi\sin\theta, \sin\varphi\sin\theta)\cdot\vec{\sigma}\right)$$

$$= 1/2\left(\mathbb{1} + \vec{r}\cdot\vec{\sigma}\right)$$

---

④ $\langle\psi|\sigma_i|\psi\rangle = \mathrm{tr}(\langle\psi|\sigma_i|\psi\rangle)$

$= \mathrm{tr}(|\psi\rangle\langle\psi|\sigma_i) = \mathrm{tr}\left(\frac{1}{2}(1+\vec{v}\cdot\vec{\sigma})\sigma_i\right) = \frac{1}{2}\mathrm{tr}(v_i\sigma_i) = v_i$

---

⑤ $|\phi\rangle\langle\phi| = 1/2\left(\mathbb{1} - \vec{v}\cdot\vec{\sigma}\right)$

$\mathrm{tr}\left((|\phi\rangle\langle\phi|)(|\psi\rangle\langle\psi|)\right) = \mathrm{tr}\left(\frac{1}{4}(1-\vec{v}\cdot\vec{\sigma})(1+\vec{v}\cdot\vec{\sigma})\right)$

$= \frac{1}{4}\mathrm{tr}\left(I - \vec{v}\cdot\vec{\sigma} + \vec{v}\cdot\vec{\sigma} - |\vec{v}|^2 I\right)$

$= 0$

Problem #03 (Bell States)

① Any 2x2 unitary can be written as

$$U = \begin{pmatrix} a & b \\ -e^{i\varphi} b^* & e^{i\varphi} a^* \end{pmatrix} \qquad |a|^2 + |b|^2 = 1.$$

$$U|0\rangle = \begin{pmatrix} a \\ -e^{i\varphi} b^* \end{pmatrix}, \quad U|1\rangle = \begin{pmatrix} b \\ e^{i\varphi} a^* \end{pmatrix}$$

$$(U \otimes U)|\psi^-\rangle = (U|0\rangle)(U|1\rangle) - (U|1\rangle)(U|0\rangle)$$

$$= \begin{pmatrix} ab \\ e^{i\varphi}|a|^2 \\ -e^{i\varphi}|b|^2 \\ a^* b^* \end{pmatrix} - \begin{pmatrix} ab \\ -e^{i\varphi}|b|^2 \\ e^{i\varphi}|a|^2 \\ a^* b^* \end{pmatrix} = e^{i\varphi} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$(U \otimes U)|\psi^-\rangle = e^{i\varphi}|\psi^-\rangle.$$

---

② $$S_{\vec{v}} = \begin{pmatrix} v_1 & \\ & -v_1 \end{pmatrix} + \begin{pmatrix} 0 & v_2 \\ v_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iv_3 \\ iv_3 & 0 \end{pmatrix}$$

$$S_{\vec{v}} = \begin{pmatrix} v_1 & v_2 - iv_3 \\ v_2 - iv_3 & -v_1 \end{pmatrix}$$

Let $|e_1\rangle = \begin{pmatrix} v_2 - iv_3 \\ 1 - v_1 \end{pmatrix}$, $|e_2\rangle = \begin{pmatrix} v_2 - iv_3 \\ -1 - v_1 \end{pmatrix}$

$$S_{\vec{v}}|e_1\rangle = +1|e_1\rangle, \quad S_{\vec{v}}|e_2\rangle = -1|e_2\rangle$$

Since $\sigma(S_{\vec{v}}) = \{1, -1\} = \sigma(Z)$.

there exist a U : $|0\rangle \xleftrightarrow{U} |e_1\rangle$.

$|1\rangle \xleftrightarrow{U} |e_2\rangle$

Let $P_{S+} = |e_1\rangle\langle e_1| \otimes I$

$P_{S-} = |e_2\rangle\langle e_2| \otimes I$

$\Big\}$ Projectors corresponding to measurement outcome along $\hat{v}$.

$|\psi^-\rangle \xrightarrow{\text{outcome } +1}$

$$p_{S+} = \langle\psi^-|P_{S+}|\psi^-\rangle = \langle\psi^-|\underbrace{U^\dagger\otimes U^\dagger(P_{S+})U\otimes U}|\psi^-\rangle$$

$$= U^\dagger\otimes U^\dagger(|e_1\rangle\langle e_1|\otimes I)U\otimes U$$

$$= |0\rangle\langle 0| \otimes I$$

$$= \langle\psi^-|(|0\rangle\langle 0|\otimes I)|\psi^-\rangle$$

$$= 1/2$$

$|\psi^-\rangle \xrightarrow{\text{outcome } -1}$

$$p_{S-} = \langle\psi^-|P_{S-}|\psi^-\rangle$$

$$= \langle\psi^-|(|1\rangle\langle 1|\otimes 1)|\psi^-\rangle$$

$$= 1/2$$

---

3. $(X\otimes I)|\psi^-\rangle = -\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = -(I\otimes X)|\psi^-\rangle$

$(Y\otimes I)|\psi^-\rangle = -\frac{i}{\sqrt{2}}(|00\rangle + |11\rangle) = -(I\otimes Y)|\psi^-\rangle$

$(Z\otimes I)|\psi^-\rangle = +\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = -(I\otimes Z)|\psi^-\rangle$

*Pairwise equal because $|\psi^-\rangle$ is Permutationally invariant up to a global phase.