Exercise Sheet #02

Problem #1 (Bloch Sphere for mixed States)

① Any ρ can be written as, $\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$.

(Exercise Sheet #1, Prob #2)

$$|\psi_n\rangle\langle\psi_n| = \frac{1}{2}(\mathbb{1} + \vec{r}_n \cdot \vec{\sigma}).$$

$$\Rightarrow \rho = \sum_n p_n |\psi_n\rangle\langle\psi_n| = \frac{1}{2}\sum_n p_n \mathbb{1} + \sum_n p_n \vec{r}_n \cdot \vec{\sigma}$$

$$= \frac{1}{2}(\mathbb{1} + \underbrace{\vec{r}}_{\sum_n p_n \vec{r}_n} \cdot \vec{\sigma})$$

② $$|\vec{r}| = \left|\sum_n p_n (\cos\varphi_n \sin\theta_n, \sin\varphi_n \sin\theta_n, \cos\theta_n)\right|$$

$$\overset{\Delta}{\leq} \sum_n p_n \left|(\cos\varphi_n \sin\theta_n, \sin\varphi_n \sin\theta_n, \cos\theta_n)\right|.$$

$$= 1$$

$\Rightarrow$ for any ρ, $|\vec{r}| \leq 1$.

$\Leftarrow$ If $|\vec{r}| \leq 1$.

$$\rho = \frac{1}{2}(\mathbb{1} + (r_x, r_y, r_z) \cdot (X, Y, Z))$$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}$$

① $\mathrm{tr}(\rho) = 1$

② $\rho = \rho^\dagger$

③ $\rho \geq 0$

Since, Eigenvalues of ρ are.

$$\lambda_{\pm} = \frac{\operatorname{tr}(\rho) \pm \sqrt{\operatorname{tr}(\rho)^2 - 4\det(\rho)}}{2}$$

$$= \frac{1 \pm \sqrt{1 - 4(1-|\vec{r}|)/4}}{2} = \left(1 \pm \sqrt{|\vec{r}|}\right)/2 \geq 0$$

①,②,③ $\Rightarrow$ for any $|\vec{r}| \leq 1$, the corresponding ρ is a density matrix

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③ $\Rightarrow$ for pure state, ~~ρ~~ $\vec{r} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$

$\Rightarrow |\vec{r}| = 1$

$\Leftarrow$ if $|\vec{r}| = 1$

any $\vec{r} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$

$\rho = \frac{1}{2}(1 + \vec{r} \cdot \vec{\sigma})$

$= |\psi\rangle\langle\psi|$ (Exercise Sheet #1, Problem 2)

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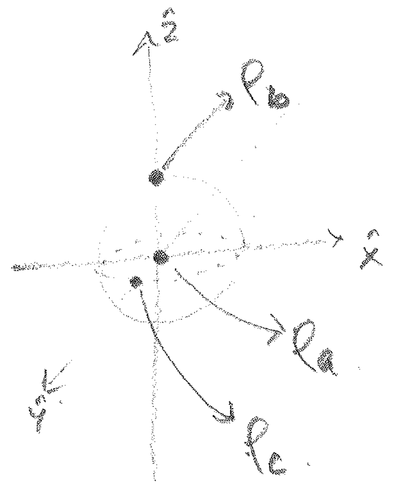
④



Problem # 02 (Measurement and filtering).

① $\Pi_0 = \left(|0\rangle\langle 0| + \sqrt{r}\,|1\rangle\langle 1|\right)_A \otimes I_B.$

$$= \begin{pmatrix} 1 & \\ & \sqrt{r} \end{pmatrix} \otimes I_B$$

$$\Pi_1 = \begin{pmatrix} 0 & \\ & \sqrt{1-r} \end{pmatrix} \otimes I_B.$$

$$\Pi_0 \Pi_0^\dagger + \Pi_1 \Pi^\dagger = \mathbb{1}_{AB}.$$

$\Rightarrow \Pi_0, \Pi_1$ define a POVM

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②

$$|\phi_x\rangle \xrightarrow{\Pi_0} \frac{\Pi_0|\phi_x\rangle}{\sqrt{\langle\phi_x|\Pi_0^\dagger \Pi_0|\phi_x\rangle}} = \frac{\sqrt{x}\,|00\rangle + \sqrt{r(1-x)}\,|11\rangle}{\sqrt{x + r(1-x)}}$$

and $p_0 = x + r(1-x)$

$$|\phi_x\rangle \xrightarrow{\Pi_1} \frac{\Pi_1|\phi_x\rangle}{\sqrt{\langle\phi_x|\Pi_1^\dagger \Pi_1|\phi_x\rangle}} = \frac{\sqrt{(1-x)(1-r)}\,|11\rangle}{\sqrt{(1-x)(1-r)}}$$

and $p_1 = (1-x)(1-r)$

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③ $$\frac{\sqrt{x}}{\sqrt{x+r(1-x)}} = \frac{\sqrt{r(1-x)}}{\sqrt{x+r(1-x)}}$$

$$\Rightarrow r = x/1-x$$

$$\Rightarrow \frac{\sqrt{x}}{\sqrt{x+r(1-x)}} = \frac{\sqrt{r(1-x)}}{\sqrt{x+r(1-x)}} = \frac{1}{\sqrt{2}}.$$

Problem # 3 (Quantum Channels)

$$\mathcal{E}(\rho) = \sum_\alpha M_\alpha \rho M_\alpha^\dagger \longrightarrow \circledast$$

Kraus reps. implies CP

$\rho \geq 0 \Rightarrow \rho = QQ^\dagger$ for some Q

Any $\rho_{AB} = \sum_{i,j} c_{ij}\, \rho_i^A \otimes \rho_j^B$

$$(I_A \otimes \mathcal{E})\rho^{AB} = \sum_{ij} c_{ij}\, \rho_i^A \sum_\alpha M_\alpha \rho_j^B M_\alpha^\dagger$$

$$= \sum_\alpha (I_A \otimes M_\alpha)\, \rho\, (I_A \otimes M_\alpha)^\dagger$$

$$= \sum_\alpha (I_A \otimes M_\alpha) Q Q^\dagger (I_A \otimes M_\alpha)^\dagger$$

$$= \sum_\alpha \underbrace{\left((I_A \otimes M_\alpha) Q\right)}_{B_\alpha} \left((I_A \otimes M_\alpha) Q\right)^\dagger$$

$$= \sum_\alpha B_\alpha B_\alpha^\dagger$$

$$\geq 0$$

① Dephasing channel.

$$\mathcal{E}(\rho) = (1-p)^{1/2} I \rho I (1-p)^{1/2} + p^{1/2} Z \rho Z p^{1/2}$$

$$M_0 = (1-p)^{1/2} I, \quad M_1 = p^{1/2} Z$$

$M_0 M_0^\dagger + M_1 M_1^\dagger = \mathbb{1} \rightarrow$ trace preserving

$\circledast \Rightarrow \mathcal{E}(\rho)$ is CPTP.

$$\mathcal{E}\left(\tfrac{1}{2}(\mathbb{1} + r_x X + r_y Y + r_z Z)\right)$$

$$= \tfrac{1}{2}(1-p)\left(I + r_x X + r_y Y + r_z Z\right) + \tfrac{1}{2} p \underbrace{Z\left(I + r_x X + r_y Y + r_z Z\right)Z}_{I - r_x X - r_y Y + r_z Z}$$

$$\mathcal{E}(\rho) = \frac{1}{2}\left(\mathbb{1} + (1-2P)\gamma_x X + (1-2P)\gamma_y Y + \gamma_z Z\right)$$

$$\Rightarrow (\gamma_x, \gamma_y, \gamma_z) \xrightarrow{\mathcal{E}} \left((1-2P)\gamma_x, (1-2P)\gamma_y, \gamma_z Z\right)$$

② Amplitude daming channel

$$\mathcal{E}(\rho) = \Pi_0 \rho \Pi_0^\dagger + \Pi_1 \rho \Pi_1^\dagger$$

$$\Pi_0 = \sqrt{\gamma}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 1 & \\ & \sqrt{1-\gamma} \end{pmatrix}$$

$\Pi_0^\dagger \Pi_0 + \Pi_1^\dagger \Pi_1 = I_{AB} \Rightarrow$ Trace Preserving

⊛ $\Rightarrow$ CP, So $\mathcal{E}(\rho)$ is CPTP.

ⓐ
$$\mathcal{E}(\rho) = \gamma\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1-P & \eta \\ \eta^* & P \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & \\ & \sqrt{1-\gamma} \end{pmatrix}\begin{pmatrix} 1-P & \eta \\ \eta^* & P \end{pmatrix}\begin{pmatrix} 1 & \\ & \sqrt{1-\gamma} \end{pmatrix}$$

$$= \begin{pmatrix} 1-P(1-\gamma) & \eta(\sqrt{1-\gamma}) \\ \eta^* \sqrt{1-\gamma} & P(1-\gamma) \end{pmatrix}$$

ⓑ
$$(\mathcal{E}_1 \circ \mathcal{E}_2)(\rho) = \mathcal{E}_1(\mathcal{E}(\rho))$$

$$= M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger$$

$$M_0 = \sqrt{\gamma_1 \gamma_2}\,|0\rangle\langle 1|, \quad M_1 = |0\rangle\langle 0| + \sqrt{(1-\gamma_1)(1-\gamma_2)}\,|1\rangle\langle 1|$$

③ Twirling operation

$$\mathcal{E}(\rho) = \tfrac{1}{4}\rho + \tfrac{1}{4} X\rho X + \tfrac{1}{4} Y\rho Y + \tfrac{1}{4} Z\rho Z$$

$M_0 = \frac{1}{2}\rho$, $M_i = \frac{1}{2}\sigma_i \xrightarrow{⊛}$ CP

Also $\sum_{\alpha=0}^{3} M_\alpha^\dagger M_\alpha = I \rightarrow$ TP

$$\mathcal{E}(\rho) = \tfrac{1}{8}(\mathbb{1} + \vec{\gamma}.\vec{\sigma}) + \tfrac{1}{8}X(\mathbb{1} + \vec{\gamma}.\vec{\sigma})\cancel{X} + \tfrac{1}{8}Y(\mathbb{1} + \vec{\gamma}.\vec{\sigma})Y + \tfrac{1}{8}Z(\mathbb{1} + \vec{\gamma}.\vec{\sigma})Z$$

$= \frac{1}{2}\mathbb{1}$ mixed state

Problem # 04 (Purification)

1 (a) $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

$$\boxed{|\psi^1\rangle_{AR} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AR} + |11\rangle_{AR}\right)}$$

$$\mathrm{tr}_R\left(|\psi^1\rangle_{AR}\langle\psi^1|_{AR}\right) = \mathrm{tr}_R\left(\frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)_{AR}\right)$$

$$= \frac{1}{2}|0\rangle_A\langle 0|_A\,\mathrm{tr}\left(|0\rangle\langle 0|\right) + \frac{1}{2}|0\rangle_A\langle 1|_A\,\mathrm{tr}\left(|0\rangle\langle 1|\right)$$

$$+ \frac{1}{2}|1\rangle_A\langle 0|_A\,\mathrm{tr}\left(|1\rangle\langle 0|\right) + \frac{1}{2}|1\rangle\langle 1|\,\mathrm{tr}\left(|1\rangle\langle 1|\right)$$

$$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|.$$

$\rho = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|.$

$$|\psi^2\rangle_{AR} = \frac{1}{\sqrt{2}}\left(|+0\rangle_{AR} + |-1\rangle_{AR}\right)$$

$\cancel{\mathrm{tr}_R\left(|\psi^2\rangle_{AR}\langle\psi^2|_{AR}\right) = \mathrm{tr}_R\left(\frac{1}{2}\left(|+0\rangle\langle +0| + |+0\rangle\langle +0|\right.\right.}$

$$\mathrm{tr}_R\left(|\psi^2\rangle_{AR}\langle\psi^2|_{AR}\right) = \mathrm{tr}_R\left(\frac{1}{2}\left(|+0\rangle\langle +0| + |+0\rangle\langle -1| + |-1\rangle\langle +0| + |-1\rangle\langle -1|\right)_{AR}\right)$$

$$= \frac{1}{2}|+\rangle\langle +|\,\mathrm{tr}\left(|0\rangle\langle 0|\right) + \frac{1}{2}|+\rangle\langle -|\,\mathrm{tr}\left(|0\rangle\langle 1|\right)$$

$$+ \frac{1}{2}|-\rangle\langle +|\,\mathrm{tr}\left(|1\rangle\langle 0|\right) + \frac{1}{2}|-\rangle\langle -|\,\mathrm{tr}\left(|1\rangle\langle 1|\right).$$

$$= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|.$$

$$\rho = \frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -|$$

$$|\psi^3\rangle_{AR} = \frac{1}{2}\left(|00\rangle + |11\rangle + |+2\rangle + |-3\rangle\right)_{AR}$$

$$\mathrm{tr}_R\left(|\psi^3\rangle_{AR}\langle\psi^3|_{AR}\right) = \frac{1}{4}|0\rangle\langle 0|\,\mathrm{tr}\left(|0\rangle\langle 0|\right) + \cdots$$

$$= \frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -|$$

Three purifications

(b) $|\psi^1\rangle_{AR} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$$|\psi^2\rangle_{AR} = \frac{1}{\sqrt{2}}\left(|+0\rangle + |-1\rangle\right) = \frac{1}{2}\left(|00\rangle + |10\rangle + |01\rangle - |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0+\rangle + |1-\rangle\right)$$

$$|\psi^3\rangle_{AR} = \frac{1}{2}\left(|00\rangle + |11\rangle + |+2\rangle + |-3\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle + |11\rangle + \frac{1}{\sqrt{2}}\left(|02\rangle + |12\rangle + |03\rangle - |13\rangle\right)\right)$$

$$= \frac{1}{2}\left(|00\rangle + |11\rangle + |0\tilde{+}\rangle + |1\tilde{-}\rangle\right)$$

$$|\tilde{+}\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle + |3\rangle\right)$$

$$|\tilde{-}\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle - |3\rangle\right)$$

$$|\psi^3\rangle_{AR} = \frac{1}{2}\left(|0\rangle\left(|0\rangle + |\tilde{+}\rangle\right) + |1\rangle\left(|1\rangle + |\tilde{-}\rangle\right)\right)$$

Let $U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$(I_A \otimes U)|\psi^1\rangle_{AR} = \frac{1}{\sqrt{2}}\left(|0+\rangle + |1-\rangle\right) = |\psi^2\rangle_{AR}$$

$$(I_A \otimes U^\dagger)|\psi^2\rangle_{AR} = |\psi^1\rangle_{AR} \Rightarrow |\psi^1\rangle_{AR} \xleftrightarrow{I\otimes U} |\psi^2\rangle_{AR}$$

Let $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

U is a one way unitary (Isometry)
$UU^\dagger = I$ but $U^\dagger U \neq I$

$(I_A \otimes U)|\psi^3\rangle_{AR} = |\psi^1\rangle_{AR}$ $\quad |\psi^1\rangle_{AR} \xleftrightarrow{I\otimes U} |\psi^3\rangle_{AR}$

$(I_A \otimes U^\dagger)|\psi^1\rangle_{AR} = |\psi^3\rangle_{AR}$

(c) Let $\Pi_0 = |0\rangle\langle 0|$, $\Pi_1 = |1\rangle\langle 1|$.
$M_1 = \{\Pi_0, \Pi_1\}$ defines a POVM.

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB} \xrightarrow[\text{B by } M_1]{\text{Measurement on}} \left\{\left(\tfrac{1}{2}, |0\rangle_A\right), \left(\tfrac{1}{2}, |1\rangle_A\right)\right\}.$$
$$\equiv \frac{1}{2}|0\rangle_A\langle 0|_A + \frac{1}{2}|1\rangle_A\langle 1|_A.$$

Let $\Pi_0 = |+\rangle\langle +|$, $\Pi_1 = |-\rangle\langle -|$.
$M_2 = \{\Pi_0, \Pi_1\}$ defines a POVM.

$$|\psi\rangle_{AB} \xrightarrow[\text{by } M_2]{\text{Measurement on B}} \left\{\left(\tfrac{1}{2}, |+\rangle_A\right), \left(\tfrac{1}{2}, |-\rangle_A\right)\right\}$$
$$\equiv \frac{1}{2}|+\rangle_A\langle +|_A + \frac{1}{2}|-\rangle_A\langle -|_A.$$

Let $\Pi_0 = 1/\sqrt{2}\,|0\rangle\langle 0|$, $\Pi_1 = \frac{1}{\sqrt{2}}|1\rangle\langle 1|$, $\Pi_2 = \frac{1}{\sqrt{2}}|+\rangle\langle +|$,
$\Pi_3 = \frac{1}{\sqrt{2}}|-\rangle\langle -|$

$M_3 = \{\Pi_0, \Pi_1, \Pi_2, \Pi_3\}$ defines a POVB.

$$|\psi\rangle_{AB} \xrightarrow[\text{by } M_3.]{\text{Measurement on B}} \left\{\left(\tfrac{1}{4}, |0\rangle_A\right), \left(\tfrac{1}{4}, |1\rangle_A\right), \left(\tfrac{1}{4}, |+\rangle_A\right), \left(\tfrac{1}{4}, |-\rangle_A\right)\right\}.$$
$$\equiv \frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -|$$

(2)

(a) $u_{ij} = \langle \phi_j | \psi_i \rangle \sqrt{p_i/q_j}$

$$\sum_j u_{ij} \sqrt{q_j} | \phi_j \rangle = \sum_j \sqrt{q_j} | \phi_j \rangle \langle \phi_j | \psi_i \rangle \sqrt{p_i/q_j}$$

$$= \sum_j | \phi_j \rangle \langle \phi_j | \psi_i \rangle \sqrt{p_i}$$

$$= \sqrt{p_i} \, | \psi_i \rangle .$$

(b) $$\sum_i u_{ij} u_{ij'}^* = \sum_i \langle \phi_j | \psi_i \rangle \langle \psi_i | \phi_{j'} \rangle p_i / \sqrt{q_j q_{j'}}$$

$$= \langle \phi_j | \underbrace{\sum_i (| \psi_i \rangle \langle \psi_i | p_i)}_{\rho} | \phi_{j'} \rangle / \sqrt{q_j q_{j'}}$$

$$= \langle \phi_j | \rho | \phi_{j'} \rangle / \sqrt{q_j q_{j'}}$$

$|\phi_j\rangle$ is ONB (orthonormal basis).

$$\Rightarrow \sum_i u_{ij} u_{ij'}^* = q_j \delta_{jj'} / \sqrt{q_j q_{j'}}$$

$$\Rightarrow (UU^\dagger)_{jj'} = \delta_{jj'}$$

$\Rightarrow$ columns of U are orthogonal.

(c) wlog, assuming that the ensemble $|\phi_i\rangle$ is small, by padding $|\phi_j\rangle$ with zero vectors it can be shown $(U^\dagger U)_{jj'} = \delta_{jj'}$

$\Rightarrow$ U forms a unitary.

(c) $|\phi_j\rangle$ does not form an eigenbasis

Let $\rho = \sum_K \lambda_K |K\rangle\langle K|$ ↳ orthonormal vectors

Let $|\phi\rangle$ be a vector orthogonal to all $|K\rangle$

$$0 = \langle\phi|\rho|\phi\rangle = \sum_j \langle\phi|\phi_j\rangle\langle\phi_j|\phi\rangle q_j$$

$$= \sum_j |\langle\phi|\phi_j\rangle|^2 q_j$$ ↳ ≥ 0

$\Rightarrow$ Every $|\phi\rangle$ orthogonal to all $|K\rangle$ is also orthogonal to all $|\phi_j\rangle$

$\Rightarrow$ $|\phi_j\rangle$ can be expressed as a linear combination of $|K\rangle$

$$|\phi_j\rangle = \sum_K c_{jK} |K\rangle$$

$$\Rightarrow \rho = \sum_\lambda \lambda_K |K\rangle\langle K| = \sum_{K\ell}\left(\sum_j c_{jK} c_{j\ell}^*\right)\sqrt{\lambda_K \lambda_\ell}\,|K\rangle\langle \ell|$$

oprators $\sqrt{\lambda_K \lambda_\ell}$ are linearly independent.

$$\Rightarrow \sum_j c_{jK} c_{j\ell}^* = \delta_{K\ell}$$

and C forms a unitary

$$\Rightarrow |\phi_j\rangle = \sum_K c_{jK} |K\rangle$$

Similarly $|\psi_i\rangle = \sum_K d_{iK} |K\rangle$

$|\phi_j\rangle$ is related to $|K\rangle$ by a unitary

$|\psi_j\rangle$ is related to $|K\rangle$ by a unitary

so the two are related by a unitary.