Exercise Sheet # 03

Problem # 01 (CHSH inequality I)

① Expectation value is, $\langle a_x b_y \rangle = \sum_{a_x, b_y} a_x b_y P(a, b | x, y)$ (* a typo in exercise sheet)

LHV Condition, $P(a, b | x, y) = \sum_\lambda p_\lambda P_\lambda^A (a|x) P_\lambda^B (b|y)$

$$\Rightarrow \langle a_x b_y \rangle = \sum_{a_x b_y} a_x b_y \sum_\lambda p_\lambda P_\lambda^A (a|x) P_\lambda^B (b|y)$$

$$= \sum_\lambda p_\lambda \Big(\sum_{a_x} a_x P_\lambda^A (a|x) \Big) \Big(\sum_{b_y} b_y P_\lambda^B (b|y) \Big)$$

$$\langle a_x b_y \rangle = \sum_\lambda p_\lambda \langle a_x \rangle_\lambda \langle b_y \rangle_\lambda \equiv \sum_\lambda p_\lambda \langle a_x b_y \rangle_\lambda \longrightarrow *$$

$$\langle C \rangle_\lambda = \langle a_0 b_0 \rangle_\lambda + \langle a_1 b_0 \rangle_\lambda + \langle a_0 b_1 \rangle_\lambda - \langle a_1 b_1 \rangle_\lambda$$

$$\overset{*}{=} \langle a_0 \rangle_\lambda \langle b_0 \rangle_\lambda + \langle a_1 \rangle_\lambda \langle b_0 \rangle_\lambda + \langle a_0 \rangle_\lambda \langle b_1 \rangle_\lambda - \langle a_1 \rangle_\lambda \langle b_1 \rangle_\lambda$$

$$= \big(\langle a_0 \rangle_\lambda + \langle a_1 \rangle_\lambda \big) \langle b_0 \rangle_\lambda + \big(\langle a_0 \rangle_\lambda - \langle a_1 \rangle_\lambda \big) \langle b_1 \rangle_\lambda$$

$$\langle b_0 \rangle_\lambda, \langle b_1 \rangle_\lambda \in \{1, -1\}$$

$$\Rightarrow |\langle C \rangle_\lambda| \leq |\langle a_0 \rangle_\lambda + \langle a_1 \rangle_\lambda| + |\langle a_0 \rangle_\lambda - \langle a_1 \rangle_\lambda|$$

wlog let $\langle a_0 \rangle_\lambda \geq \langle a_1 \rangle_\lambda$

$$\Rightarrow |\langle C \rangle_\lambda| \leq 2 |\langle a_0 \rangle_\lambda|$$

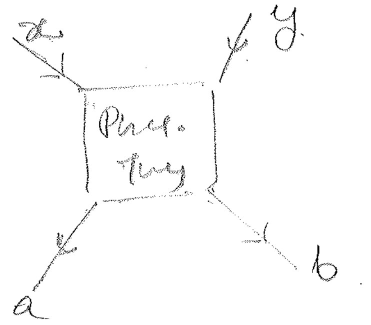
$$|\langle C \rangle_\lambda| \leq 2$$

$$\boxed{\langle C \rangle = \sum_\lambda p_\lambda |\langle C \rangle_\lambda| \leq 2}$$

$P(a, b | x, y)$ is seperable allowed for this bound.

(2)

A Phy. theory (black box) is "non-signaling" if the output a does not tell anything about the input of y, and vice-versa.



$$P^A(a|x) = \sum_b P(a,b|x,y) = \sum_b P(a,b|x,y') \rightarrow (*)$$

Consider a non-signaling theory called PR-box (Popescu, Rohrlich)

$$P(a,b|x,y) = \begin{cases} 1/2 & \text{if } a \oplus b = x \cdot y \\ 0 & \text{otherwise} \end{cases} \rightarrow (**)$$

Exclusive OR

Logical And

$P(0,0|0,0) = 1/2$

$P(1,1|0,0) = 1/2$

$P(0,0|0,1) = 1/2$

$P(1,1|0,1) = 1/2$

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$P(0,1|1,1) = 1/2$

$P(1,0|1,1) = 1/2$

PR-box is non-signaling as (*) holds for $P(a,b|x,y)$ in (**)

Since

$$\langle a_x b_y \rangle = P(0,0|x,y) - P(0,1|x,y) - P(1,0|x,y) + P(1,1|x,y)$$

for the PR box

$$\langle a_0 b_0 \rangle = \tfrac{1}{2} - 0 - 0 + \tfrac{1}{2} = 1$$

$$\langle a_1 b_0 \rangle = \tfrac{1}{2} - 0 - 0 + \tfrac{1}{2} = 1$$

$$\langle a_1 b_0 \rangle = \tfrac{1}{2} - 0 - 0 + \tfrac{1}{2} = 1.$$

$$\langle a_1 b_1 \rangle = 0 - \tfrac{1}{2} - \tfrac{1}{2} + 0 = -1$$

$$\Rightarrow \langle C \rangle = 1 + 1 + 1 - (-1) = 4.$$

& since $|\langle a_x b_y \rangle| \leq 1$.
the algebraic bound on
$|\langle C \rangle| \leq 4$.

$\Rightarrow$ PR-box (non-signaling distribution) allows for a maximum possible value of $\langle C \rangle$.

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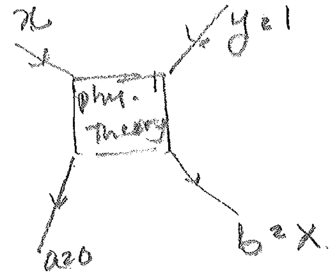
(3) A theory is signaling if it is not non-signaling



$$P(a,b|x,y) = \begin{cases} 1 & \text{if } a=0 \text{ and } b=x \cdot y. \\ 0 & \text{otherwise.} \end{cases}$$

output to is completely determined by x.

Problem #02 (Tsirelson's bound)

① $C = a_0b_0 + a_1b_0 + a_0b_1 - a_1b_1$

$C^2 = a_0b_0a_0b_0 + a_0b_0a_1b_0 + a_0b_0a_0b_1 - a_0b_0a_1b_1$
$+ a_1b_0a_0b_0 + a_1b_0a_1b_0 + a_1b_0a_0b_1 - a_1b_0a_1b_1$

$+ \cdot \;\cdot\; \cdot\; \cdot$

$= 4I - a_0a_1b_0b_1 + a_1a_0b_0b_1 + a_0a_1b_1b_0 - a_1a_0b_1b_0$

$= 4I + a_0a_1(b_1b_0 - b_0b_1) - a_1a_0(b_1b_0 - b_0b_1)$

$C^2 = 4I + (a_0a_1 - a_1a_0)(b_1b_0 - b_0b_1)$

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② $\|m\| = \max_{|\psi\rangle} \dfrac{\|m|\psi\rangle\|}{\||\psi\rangle\|}$

ⓐ $\|mn\| = \max_{|\psi\rangle} \dfrac{\|mn|\psi\rangle\|}{\||\psi\rangle\|}$

$= \max_{|\psi\rangle} \dfrac{\|mn|\psi\rangle\|}{\|n|\psi\rangle\|} \dfrac{\|n|\psi\rangle\|}{\||\psi\rangle\|}$

$\leq \left(\max_{|\psi\rangle} \dfrac{\|mn|\psi\rangle\|}{\|n|\psi\rangle\|}\right)\left(\max_{|\psi\rangle} \dfrac{\|n|\psi\rangle\|}{\||\psi\rangle\|}\right)$

$= \left(\max_{|\phi\rangle} \dfrac{\|m|\phi\rangle\|}{\||\phi\rangle\|}\right)\left(\max_{|\psi\rangle} \dfrac{\|n|\psi\rangle\|}{\||\psi\rangle\|}\right)$

$= \|m\|\,\|n\|$

(b) $$\|M+N\| = \max_{|\psi\rangle} \frac{\|(M+N)|\psi\rangle\|}{\||\psi\rangle\|} \leq \max_{|\psi\rangle} \frac{\|M|\psi\rangle\| + \|N|\psi\rangle\|}{\||\psi\rangle\|}$$

$$\leq \max_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} + \max_{|\psi\rangle} \frac{\|N|\psi\rangle\|}{\||\psi\rangle\|}$$

$$= \|M\| + \|N\|.$$

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(3) $$\|C^2\| = \|4I + (a_0a_1 - a_1a_0)(b_1b_0 - b_0b_1)\|$$

$$\leq 4\|I\| + \|a_0a_1 - a_1a_0\| \, \|b_1b_0 - b_0b_1\|$$

$$\leq 4 + (\|a_0a_1\| + \|a_1a_0\|)(\|b_1b_0\| + \|b_0b_1\|)$$

$$\leq 4 + 2\times 2$$

$$\Rightarrow \boxed{\|C^2\| \leq 8}$$

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(4) $$\|C^2\| = \|CC\| = \|C^\dagger C\| := \|C\|^2$$

$$\|C^2\| \leq 8$$

$$\|C\|^2 \leq 8$$

$$\|C\| \leq 2\sqrt{2}.$$

Problem # 03 (Remote State Prepration)

$$|\psi\rangle = (|0\rangle + e^{i\alpha}|1\rangle)/\sqrt{2}$$

$$|\psi^{\perp}\rangle = (|0\rangle - e^{i\alpha}|1\rangle)/\sqrt{2}$$

$$\langle\psi|\psi^{\perp}\rangle = 0$$

$$|\phi^{+}\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

Let $M_0 = |\psi\rangle\langle\psi|$, $M_1 = |\psi^{\perp}\rangle\langle\psi^{\perp}|$

$M = \{M_0, M_1\}$ defines a measurement on Alice qubit

If the outcome is "0", then the post measurement State is, upto a normalization factor.

$$(M_0 \otimes I_B)|\phi^{+}\rangle_{AB} = (|\psi\rangle_A\langle\psi|_A \otimes I_B)(|00\rangle_{AB} + |11\rangle_{AB})$$

$$= \left((|0\rangle + e^{i\alpha}|1\rangle)(\langle 0| + e^{-i\alpha}\langle 1|) \otimes I_B\right)(|00\rangle + |11\rangle)$$

$$= (|\psi\rangle)(|0\rangle + e^{-i\alpha}|1\rangle)$$

$$= (|\psi\rangle)\, e^{-i\alpha}(e^{i\alpha}|0\rangle + |1\rangle)$$

$$= (|\psi\rangle\, e^{-i\alpha} X (|0\rangle + e^{i\alpha}|1\rangle)$$

$$= |\psi\rangle_A\, e^{-i\alpha} X |\psi\rangle_B \Rightarrow \text{apply } X \text{ on Bob Side.}$$

If the outcome is "1" then the post measurement state is, up to normalization factor

$$(M_1 \otimes I_B)|\phi^+\rangle_{AB} = (|\psi^\perp\rangle_A\langle\psi^\perp|_A \otimes I_B)\,|\phi^+\rangle_{AB}$$

$$= \big((|0\rangle - e^{i\alpha}|1\rangle)(\langle 0| - e^{-i\alpha}\langle 1|) \otimes I\big)(|00\rangle + |11\rangle)$$

$$= |\psi^\perp\rangle_A\,(|0\rangle - e^{-i\alpha}|1\rangle)$$

$$= |\psi^\perp\rangle_A\, e^{-i\alpha}(e^{i\alpha}|0\rangle - |1\rangle)$$

$$= |\psi^\perp\rangle_A\, e^{-i\alpha}\, ZX\,|\psi^\perp\rangle$$ apply ZX on Bob side.

Problem # 04 (LOCC Protocol)

a. $M_j = \sum_{k,l} m_{j,kl} |k\rangle_B \langle l|_B$

$N_j = \sum_{k,l} m_{j,kl} |k\rangle_A \langle l|_A$

$|\psi\rangle = \sum \lambda_l |l\rangle_A |l_B\rangle$

$|\psi_j\rangle = M_j |\psi\rangle = \sum_{k,l} m_{j,kl} |k\rangle_B \langle l|_B \sum_i \lambda_i |i\rangle_A |i\rangle_B$

$= \sum_{k,l,i} \lambda_i m_{j,kl} |k\rangle_B |i\rangle_A \delta_{li} = \sum_{k,l} m_{j,kl} |l\rangle_A |k\rangle_B \lambda_l$

$= \sum_{k,l} \tilde{m}_{j,kl} |l\rangle_A |k\rangle_B$

Let $\sigma(\tilde{m}_j)$ be the singular values of $\tilde{m}_j$

$\Rightarrow \sigma(\tilde{m}_j)$ are the Schmidt cofficients of $|\psi_j\rangle$

$|\phi_j\rangle = \sum_{k,l} \tilde{m}_{j,kl} |k\rangle_A |l\rangle_B$

$= \sum_{k,l} \tilde{m}^T_{j,kl} |l\rangle_A |k\rangle_B$

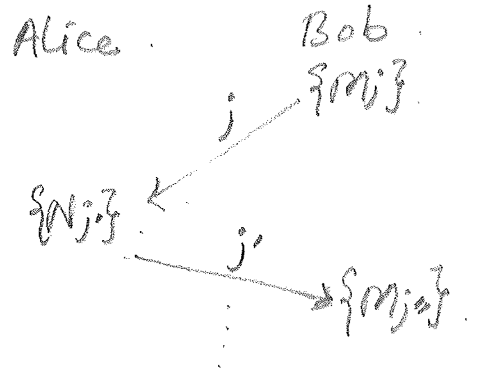
Let $\sigma(\tilde{m}_j^T)$ be the singular values of $\tilde{m}_j^T$ be Schmidt cofficients of $|\phi_j\rangle$.

$\Rightarrow \sigma(\tilde{m}_j^T) = \sigma(\tilde{m}_j)$ are Schmidt cofficients

$|\psi_j\rangle$ and $|\phi_j\rangle$ have same Schmidt Cofficient and the two states with the same Schmidt cofficient are related to each by local unitary transformation (Lec # 2, Pg. 28).

$\Rightarrow (u_j \otimes v_j)|\psi_j\rangle = |\phi_j\rangle$

b.



$$\cdots M_{j'''}^{A} M_{j''}^{B} M_{j'}^{A} M_{j}^{B} |\psi\rangle$$

$$= \cdots \left(U_{j'''}^{A} \otimes V_{j'''}^{B}\right)\left(I^{A} \otimes M_{j''}^{B}\right)\left(U_{j'}^{A} \otimes V_{j'}^{B}\right)\left(I^{A} \otimes M_{j}^{B}\right)|\psi\rangle$$

$$= \left(\cdots U_{j'''}^{A} U_{j'}^{A}\right) \otimes \left(\cdots V_{j'''}^{B} M_{j''}^{B} V_{j'}^{B} M_{j}^{B}\right)|\psi\rangle$$

Problem #05 (Majorization).

1

(a) $x \prec y$ iff $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad 1 \leq k < d$.

$$\sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i$$

'$\prec$' is clearly reflexive by definition.

(b) If $x \prec y$ and $y \prec z$

$$\Rightarrow \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} z_i$$

and $\sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i = \sum_{i=1}^{d} z_i$

$$\Rightarrow x \prec z$$

(c) If $x \prec y$ and $y \prec x$

$$\Rightarrow \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i$$

and $\sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i$

can only be true if $x = y$.

2. If $\sum_{i=1}^{d} x_i \neq \sum_{i=1}^{d} y_i \Rightarrow x \not\prec y$ and $y \not\prec x$.

3. $x \prec y$

$$\Rightarrow x = \sum_i p_i P_i y \qquad F(x) := \sum_{i=1}^{d} f(x_i).$$

$$F(x) = F\left(\sum_i p_i P_i y\right) \overset{\text{Jensen's inequality}}{\leq} \sum_i p_i F(P_i y) = \sum_i p_i F(y) = F(y).$$

$$\Rightarrow F(x) \leq F(y).$$

④ $\rho_B^\psi = tr_A(|\psi\rangle\langle\psi|)$

If $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$

$\Rightarrow \underbrace{\lambda(\rho_B^\psi)}_{\text{Eigenvalues of } \rho_B^\psi} \prec \lambda(\rho_B^\phi)$ (N&C, Pg. # 576, Th. # 12.15)

$$S(\rho_B^\psi) = \sum_i \lambda_i(\rho_B^\psi)\log(\lambda_i(\rho_B^\psi))$$

$$= \sum_i f(\lambda_i(\rho_B^\psi)) \underset{\binom{\text{using}}{3}}{\leq} \sum_i f(\lambda_i(\rho_B^\phi))$$

$$\Rightarrow S(\rho_B^\psi) \leq S(\rho_B^\phi)$$

⑤ (See An introduction to Majorization and its Applications to Quantum Mechanics, Michael A Neilsen, Pg. # 14)

⑥ Let $x_i \prec y$, where $x_i \in x$.

$$\Rightarrow \sum_{j=1}^{d} \max(x_{ij} - t, 0) \leq \sum_{j=1}^{d} \max(y_j - t, 0) \quad \binom{\text{using}}{5}$$

x_i forms a convex set if

$$\sum_i \lambda_i x_i \prec y \quad \text{with } \sum_i \lambda_i = 1$$

$$\sum_j \max\left(\sum_i \lambda_i x_{ij} - t, 0\right) = \sum_j \max\left(\sum_i \lambda_i (x_{ij} - t, 0\right)$$

$$\leq \sum_{i,j} \max(\lambda_i(x_{ij} - t), 0) = \sum_i \lambda_i \sum_j \max(x_{ij} - t, 0)$$

$$\leq \sum_i \lambda_i \sum_{j=1}^{d} \max(y_j - t, 0) = \sum_{j=1}^{d} \max(y_j - t, 0)$$

$$\Rightarrow \sum_i \lambda_i x_i \prec y.$$