Problem # 01

Decay of Entanglement

① $\rho(t) = \begin{pmatrix} P_+ & 0 & 0 & e^{-t/T_2} \\ 0 & P_- & 0 & 0 \\ 0 & 0 & P_- & 0 \\ e^{-t/T_2} & 0 & 0 & P_+ \end{pmatrix}$

② $\rho(t)^{T_B} = P_+ |00\rangle\langle 00| + P_- |01\rangle\langle 01| + P_- |10\rangle\langle 10| + P_+ |11\rangle\langle 11|$

$+ \frac{e^{-t/T_2}}{2} |01\rangle\langle 10| + \frac{e^{-t/T_2}}{2} |10\rangle\langle 01|$

$= \begin{pmatrix} P_+ & 0 & 0 & 0 \\ 0 & P_- & e^{-t/T_2}/2 & 0 \\ 0 & e^{-t/T_2}/2 & P_- & 0 \\ 0 & 0 & 0 & P_+ \end{pmatrix}$

③ $\lambda_1 = \lambda_2 = P_+$

$\lambda_3 = P_- + \frac{1}{2} e^{-t/T_2} = \frac{1}{4} - \frac{1}{4} e^{-t/T_1} + \frac{1}{2} e^{-t/T_2}$

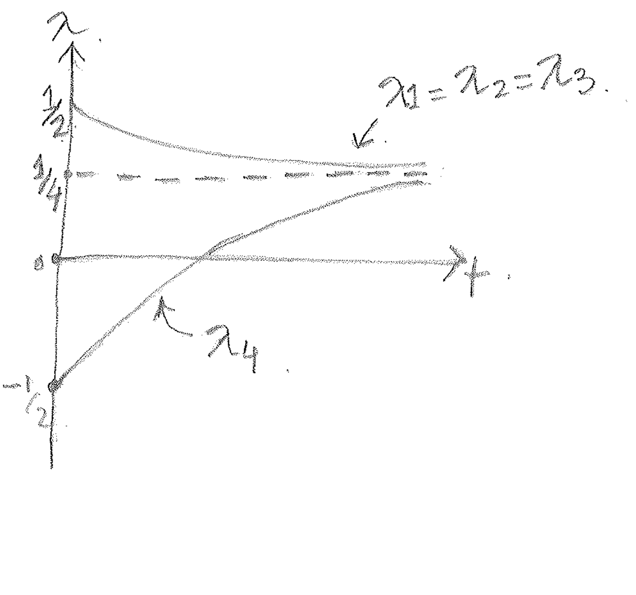
$\lambda_4 = P_- - \frac{1}{2} e^{-t/T_2} = \frac{1}{4} - \frac{1}{4} e^{-t/T_1} - \frac{1}{2} e^{-t/T_2}$

④ $T_1 = T_2 = 1$

$\lambda_1 = \lambda_2 = \frac{1}{4} + e^{-t}/4$

$\lambda_3 = \frac{1}{4} + e^{-t}/4$

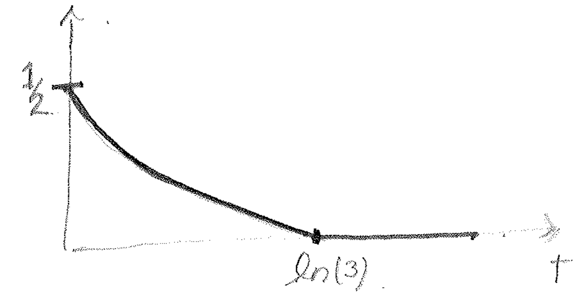
$\lambda_4 = \frac{1}{4} - \frac{3}{4} e^{-t}$



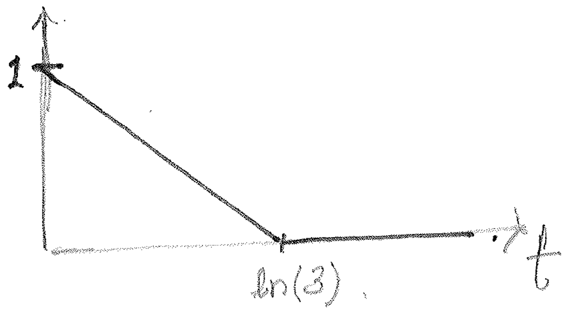
$$N(\rho(t)) = \frac{1}{2}\left(\|\rho^{T_A}\|_1 - 1\right).$$

$$\|\rho^{T_A}\|_1 = \frac{3}{4}|1+e^{-t}| + \left|\frac{1}{4} - \frac{3}{4}e^{-t}\right|$$

$$N(\rho(t)) = \frac{1}{2}\left(\frac{3}{4}|1+e^{-t}| + \left|\frac{1}{4} - \frac{3}{4}e^{-t}\right| - 1\right)$$



$$E_N(\rho(t)) = \log_2\left(\frac{3}{4}|1+e^{-t}| + \left|\frac{1}{4} - \frac{3}{4}e^{-t}\right|\right)$$



(5) At $t=0$ PPT (Positive Partial transpose).

$\lambda_4(0) < 0 \Rightarrow \rho(t)^{T_B} \ngeq 0$

$\Rightarrow \rho(0)$ is Entangled.

Since PPT Criterion is necessary and sufficient $2\otimes 2$ and $2 \otimes 3$ cases.

$\lambda_4(t) = \frac{1}{4} - \frac{3}{4}e^{-t} \geq 0$

$\Rightarrow t \geq \ln(3)$.

$\Rightarrow$ State is seperable for $t \geq \ln(3)$.

Problem # 02.

Bell inequalities and witnesses.

① For every seperable state ρ,

$$|tr(C\rho)| \leq 2. \quad \text{(CHSH inequality)} \to (*)$$

W is an entanglement witness if

$$tr(W\rho) \geq 0.$$

seperable states

Let $W = 2I - C$

$$\Rightarrow tr(W\rho) = 2\,tr(\rho) - tr(C\rho) \underset{(*)}{\geq} 0$$

② $tr(W\rho(\lambda)) = tr(2I\rho(\lambda)) - tr(C\rho(\lambda))$

$$= 2 - tr(C\rho(\lambda))$$

$$\rho(\lambda) = \lambda|\psi^-\rangle\langle\psi^-| + \frac{(1-\lambda)}{4}I \quad \text{(werner state)}$$

$$tr(W\rho(\lambda)) = 2 - \lambda\,tr(C|\psi^-\rangle\langle\psi^-|) - \frac{(1-\lambda)}{4}\underbrace{tr(C)}_{0}$$ made from pauli's

$$= 2 - \lambda\langle\psi^-|C|\psi^-\rangle$$

But $C = \sqrt{2}(\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z)$

$n_0 = (1,0,0)$
$n_1 = (1/\sqrt{2}, 0, 1/\sqrt{2})$
$n_2 = (0,0,-1)$
$n_3 = (1/\sqrt{2}, 0, -1/\sqrt{2}$

$$\Rightarrow tr(W\rho(\lambda))$$

$$= 2 - \frac{\lambda\sqrt{2}}{2}\left(\langle 01| - \langle 10|\right)\left(|10\rangle - |01\rangle - |01\rangle + |10\rangle\right)$$

$$= 2 - \frac{\lambda 2}{\sqrt{2}}\left(\langle 01| - \langle 10|\right)\left(|10\rangle - |01\rangle\right)$$

$\mathrm{tr}(W\rho(x)) = 2 + 2\sqrt{2}\,x$

State is entangled if $\mathrm{tr}(W\rho(x)) < 0$

$\Rightarrow 2 + 2\sqrt{2}\,x < 0$

$$\boxed{x < -1/\sqrt{2}}$$

Eigenvalues of $\rho(x)$ are.

$x_1 = x_2 = x_3 = \frac{1-x}{4}$, $x_0 = x + \frac{1-x}{4} = \frac{3x+1}{4}$

For $\rho(x)$ to be a density matrix

$\frac{1-x}{4} \geq 0$ and $\frac{3x+1}{4} \geq 0$

$\Rightarrow x \leq 1$ and $x \geq -1/3$

$x \in [-1/3, 1]$

Since $-1/\sqrt{2} < -\frac{1}{3} \Rightarrow W$ cannot detect entanglement in $\rho(x)$

$\mathbb{F}$ can detect entanglement in werner State

Problem # 03

Witnesses and Reduction Criterion

① $W = \mathbb{I} - d|\Omega\rangle\langle\Omega|$, $\rho = \sum_i p_i \rho^{i,A} \otimes \rho^{i,B}$

$$tr(W\rho) = tr(\rho) - d\, tr(|\Omega\rangle\langle\Omega|\rho)$$
$$= 1 - d\langle\Omega|\rho|\Omega\rangle$$
$$= 1 - \frac{d}{d}\sum_{i,j,k} p_i \langle jj|\rho^{i,A} \otimes \rho^{i,B}|kk\rangle$$
$$= 1 - \sum_{i,j,k} p_i \langle j|\rho^{i,A}|k\rangle\langle j|\rho^{i,B}|k\rangle$$
$$= 1 - \sum_i p_i\, tr(\rho^{i,AT}\rho^{i,B})$$
$$\geq 1 - \sum_i p_i \|\rho^{i,AT}\|_1 \|\rho^{i,B}\|_\infty$$
$$\geq 0$$

② $\rho_{iso}(x) = \frac{x}{d^2}\mathbb{I} + (1-x)|\Omega\rangle\langle\Omega|$

$$\rho_{iso}(x)|\Omega\rangle = \left(\frac{x}{d^2} + (1-x)\right)|\Omega\rangle = \frac{x(1-d^2)+d^2}{d^2}|\Omega\rangle$$ $|\psi_0\rangle$

Let $|\psi_i\rangle \perp |\Omega\rangle$ for $i = 1, 2, \ldots d^2-1$

$$\rho_{iso}|\psi_i\rangle = \frac{x}{d^2}|\psi_i\rangle$$

$\Rightarrow \rho_{iso}(x) \geq 0$ iff $\frac{x}{d^2} \geq 0$ and $\frac{x(1-d^2)+d^2}{d^2} \geq 0$

$$\Leftrightarrow x \geq 0 \text{ and } x \leq \frac{d^2}{d^2-1}$$

So $x \in [0, d^2/d^2-1]$

$$tr(W\rho_{iso}(x)) = tr\left((\mathbb{I} - d|\Omega\rangle\langle\Omega|)\rho_{iso}(x)\right)$$
$$= 1 - d\langle\Omega|\rho_{iso}(x)|\Omega\rangle = 1 - \frac{d}{d^2}x - d(1-x)$$

$= (d-d^2+\lambda(d^2-1))/d < 0$

iff $\quad \lambda(d-1) < d^2-d$

$\lambda < \dfrac{d^2-d}{d^2-1}$

$\Rightarrow W$ detect $\rho(\lambda)$ is for $\lambda \in [0, \frac{d}{d+1}]$.

③ $d=2$, $W = \mathbb{I} - 2|\phi^+\rangle\langle\phi^+|$, $\rho = |\psi^-\rangle\langle\psi^-|$

$\mathrm{tr}(W|\psi^-\rangle\langle\psi^-|) = \langle\psi^-|W|\psi^-\rangle$

$= \langle\psi^-|(\mathbb{I}-2|\phi^+\rangle\langle\phi^+|)|\psi^-\rangle$

$= 1 > 0.$

W does not detect if $\rho = |\psi^-\rangle\langle\psi^-|$ is entangled.

④ $\Lambda(X) = \mathrm{tr}_A(W(X_A \otimes I_B))^T$

$= \mathrm{tr}_A((\mathbb{I} - d|\Omega\rangle\langle\Omega|)(X_A \otimes I_B))^T$

$= \mathrm{tr}_A(X_A \otimes I_B)^T - \mathrm{tr}_A(\sum_{i,j} |ii\rangle\langle jj|(X_A \otimes I_B))^T$

$= \mathbb{I}\,\mathrm{tr}(X) - \left(\sum_{i',i,j} \langle i'|i\rangle\langle j|X|i'\rangle|i\rangle\langle j|\right)^T$

$= \mathbb{I}\,\mathrm{tr}(X) - \left(\sum_{i,j} \langle j|X|i\rangle|i\rangle\langle j|\right)^T$

$= \mathbb{I}\,\mathrm{tr}(X) - \sum_{i,j} \langle j|X|i\rangle|j\rangle\langle i|.$

$\Lambda(X) = \mathbb{I}\,\mathrm{tr}(X) - (X).$

$X \geq 0 \Rightarrow \{\lambda_i, |\psi_i\rangle\}$ be the eigenvalue decomposition

$\lambda_i \geq 0$

$\Rightarrow \Lambda(X) = \left(\sum_i |\psi_i\rangle\langle\psi_i|\right)\left(\sum_j \lambda_j\right) - \sum_i \lambda_i|\psi_i\rangle\langle\psi_i|$

$$= \sum_i |\psi_i\rangle\langle\psi_i| \Big(\underbrace{\sum_j x_j - x_i}_{\geq 0}\Big)$$

$$\Rightarrow \Lambda(X) \geq 0$$

(5) $$(\Lambda\otimes\mathbb{I})(\rho_{iso}(x)) = (\Lambda\otimes\mathbb{I})\left(x\,\mathbb{I}/d^2 + (1-x)|\Omega\rangle\langle\Omega|\right)$$

$$= (\Lambda\otimes\mathbb{I})\left(\frac{x}{d^2}\,\mathbb{I}_A\otimes\mathbb{I}_B + \frac{(1-x)}{d}\sum_{i,j}|i\rangle\langle j|\otimes|i\rangle\langle j|\right)$$

$$= \frac{x}{d^2}\Lambda(\mathbb{I}_A)\otimes\mathbb{I}_B + \frac{(1-x)}{d}\sum_{i,j}\Big(\Lambda(|i\rangle\langle j|)\otimes|i\rangle\langle j|\Big).$$

$$= \frac{x}{d^2}\Big(\mathbb{I}_A\operatorname{tr}(\mathbb{I}_A) - \mathbb{I}_A\Big)\otimes\mathbb{I}_B + \frac{(1-x)}{d}\sum_{i,j}\Big(\mathbb{I}_A\operatorname{tr}(|i\rangle\langle j|) - |i\rangle\langle j|\Big)\otimes|i\rangle\langle j|$$

$$= \frac{x}{d^2}(d-1)\,\mathbb{I} + \frac{(1-x)}{d}\sum_{i,j}\Big(\delta_{ij}\,\mathbb{I}_A\otimes|i\rangle\langle j| - |i\rangle\langle j|\otimes|i\rangle\langle j|\Big)$$

$$= \frac{x}{d^2}(d-1)\,\mathbb{I} + \frac{(1-x)}{d}\,\mathbb{I} - (1-x)|\Omega\rangle\langle\Omega|.$$

Eigenvalues of $(\Lambda\otimes\mathbb{I})(\rho_{iso}(x)) = \left\{\frac{x(d-1)+d(1-x)}{d^2},\ \frac{x(d-1)+d(1-x)-(1-x)d^2}{d^2}\right\}$

$\not\geq 0$ iff $x(d-1)+d(1-x) < 0$

or $x(d-1)+d(1-x)-(1-x)d^2 < 0$

$\Leftrightarrow$ $x > d$ or $x < \frac{d^2-d}{d^2-1}$

But for $\rho_{iso}(x)$ to be ≥ 0 $x \in \left[0, \frac{d^2}{d^2-1}\right]$.

$\Rightarrow$ Λ detect $\rho_{iso}(x)$ is entangled

iff $x \in \left[0, \frac{d^2-d}{d^2-1}\right]$

$$(\Lambda \otimes I)(|\psi^-\rangle\langle\psi^-|) = \frac{1}{2}\Big(\Lambda(|0\rangle\langle 0|)\,|1\rangle\langle 1| - \Lambda(|0\rangle\langle 1|)\,|1\rangle\langle 0|$$
$$- \Lambda(|1\rangle\langle 0|)\,|0\rangle\langle 1|$$
$$+ \Lambda(|1\rangle\langle 1|)\,|0\rangle\langle 0|\Big)$$

$$= \frac{1}{2}\Big(|1\rangle\langle 1| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1|$$
$$+ |0\rangle\langle 0| \otimes |0\rangle\langle 0|\Big)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \not\geq 0 \Rightarrow \Lambda \text{ detect entanglement in } |\psi^-\rangle\langle\psi^-|$$

Problem # 4

① $\exp(i\phi U) = \sum_{K=0}^{\infty} \frac{1}{K!} (i\phi U)^K$

$$= \sum_{K=0}^{\infty} \frac{i^{2K}}{(2K)!} \phi^{2K} (U^2)^K + \sum_{K=0}^{\infty} \frac{i^{(2K+1)}}{(2K+1)!} \phi^{(2K+1)} U^{2K+1}$$

$$= \left(\sum_{K=0}^{\infty} \frac{(-1)^K}{(2K)!} \phi^{2K} \right) \mathbb{1} + \left(\sum_{K=0}^{\infty} \frac{(-1)^K}{(2K+1)!} \phi^{(2K+1)} \right) i U$$

$$= \cos\phi \, \mathbb{1} + i \sin\phi \, U$$

② $R_z(\phi) = e^{i\phi Z}$

$$\rho' = R_z(\phi) \, \rho \, R_z^{\dagger}(\phi)$$

$$= (\cos\phi \, I + i\sin\phi \, Z) \, \rho \, (\cos\phi \, I - i\sin\phi \, Z)$$

$\vdots$

$$= \frac{1}{2} \Big(I + (\cos 2\phi \, \gamma_x - i\sin 2\phi \, \gamma_y) x + (\cos 2\phi \, \gamma_y + i\sin 2\phi \, \gamma_x) y + \gamma_z Z \Big)$$

$$\vec{\gamma}' = \left(\begin{array}{cc|c} \cos 2\phi & -i\sin 2\phi & \\ i\sin 2\phi & \cos 2\phi & \\ \hline & & 1 \end{array} \right) \vec{\gamma}$$

z-component of Bloch vector $\vec{\gamma}$ is invariant

Problem #05

using ④

$U = R_x(\alpha) R_z(\beta) R_x(\gamma)$

Let $A = R_x(\alpha) R_z(\beta/2)$

$B = R_z(-\beta/2) R_x\left(-\frac{\gamma+\alpha}{2}\right)$

$C = R_x\left(\frac{\gamma-\alpha}{2}\right)$

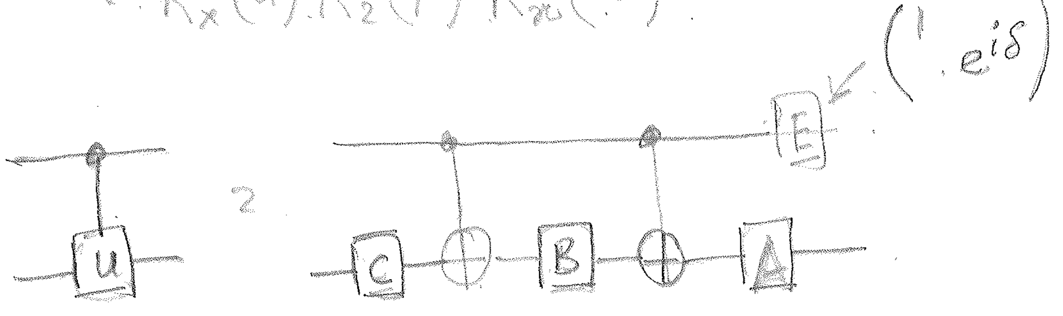
① $ABC = R_x(\alpha) R_z(\beta/2) R_z(-\beta/2) R_x\left(-\frac{\gamma+\alpha}{2}\right) R_x\left(\frac{\gamma-\alpha}{2}\right)$

$= R_x(\alpha) R_z(0) R_x(-\alpha)$

$= I$

② $A X B X C = R_x(\alpha) \; R_z(\beta/2) X R_z(-\beta/2) R_x\left(-\frac{\gamma+\alpha}{2}\right) X R_x\left(\frac{\gamma-\alpha}{2}\right)$

$= R_x(\alpha) R_z(\beta) R_x(\gamma) = U$



(3) Every 2×2 unitary is of the form

$$U = \begin{pmatrix} a & ib \\ ib^* & a^* \end{pmatrix}$$

rows and columns are orthonormal

$|a|^2 + |b|^2 = 1$.

Alternatively,

$$U = \begin{pmatrix} \cos\beta\, e^{i\psi_1} & i\sin\beta\, e^{i\psi_2} \\ i\sin\beta\, e^{-i\psi_2} & \cos\beta\, e^{-i\psi_1} \end{pmatrix}$$

Let $\psi_1 = \alpha + \gamma$

$\psi_2 = \alpha - \gamma$

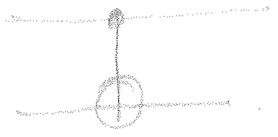
$$\Rightarrow U = \begin{pmatrix} \cos\beta\, e^{i(\alpha+\gamma)} & i\sin\beta\, e^{-i(\alpha-\gamma)} \\ i\sin\beta\, e^{-i(\alpha-\gamma)} & \cos\beta\, e^{i(\alpha+\gamma)} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} \cos\beta & i\sin\beta \\ i\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}$$

$$= R_z(\alpha)\, R_x(\beta)\, R_z(\gamma).$$

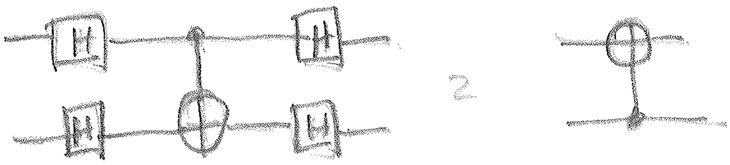
Problem # 06.

(1)



$= CNOT$.

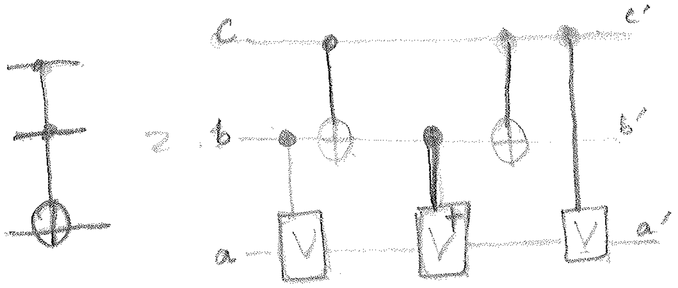
$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$(H \otimes H)\,CNOT\,(H \otimes H)$.

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(2)



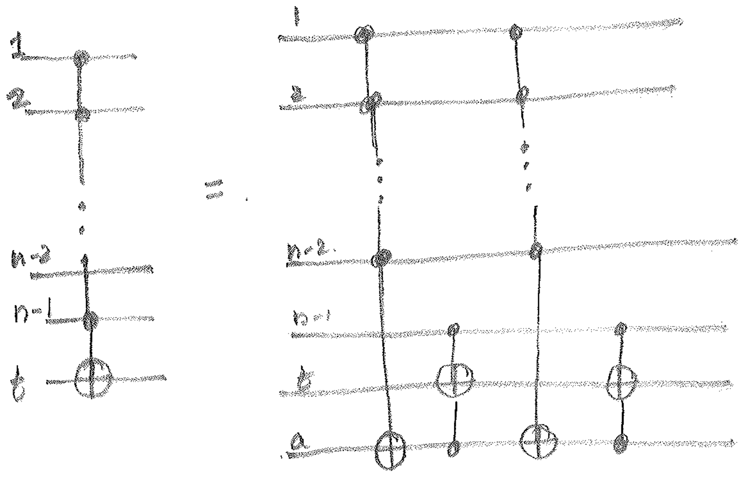
$$V = \frac{1-i}{2}(I + iX)$$

$$V^2 = X.$$

Input a	b	c.	output a'	b'	c'
0	0	0	0	0	0
1	0	0	1	0	0.
0	0	1	0	0	1
1	0	1	1	0	1.
0	1	0	0	1	0.
1	1	0.	1	1	0
0	1	1	1	1	1
1	1	1	0	1	1

Problem # 07: arxiv: quant-ph/9503016

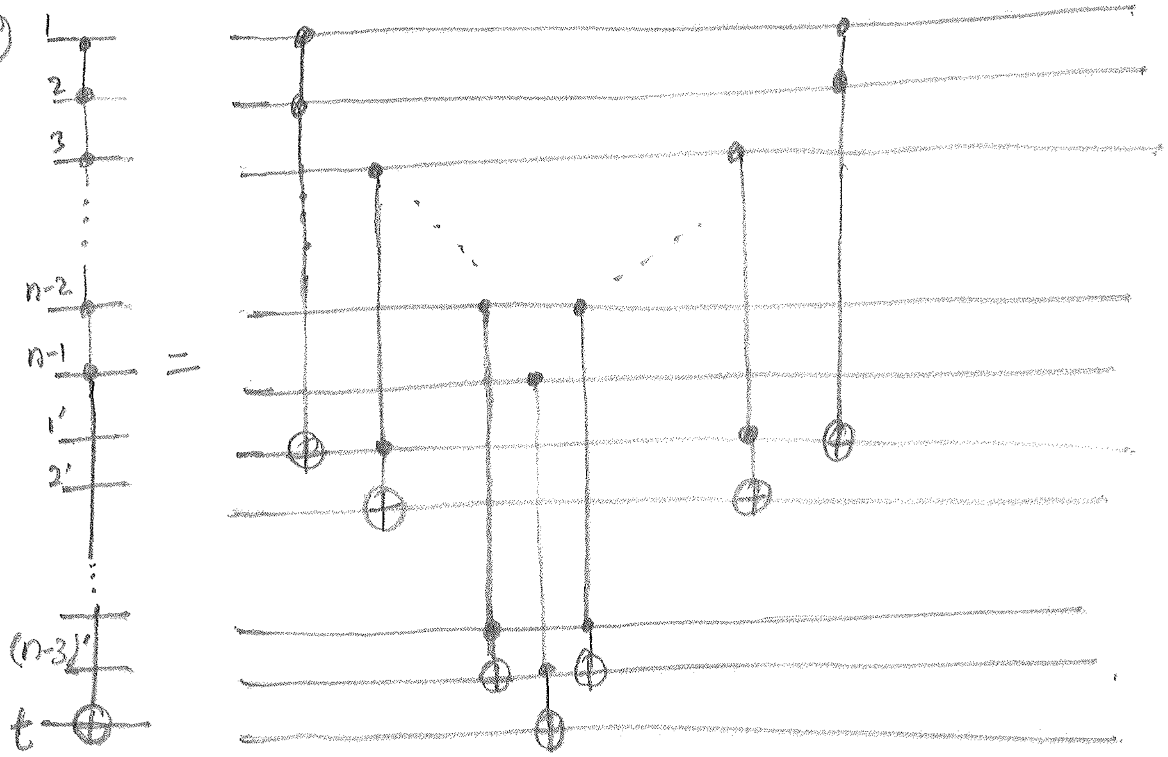
①



②
$$C(n) = 2C(n-1) + 2$$
$$C(3) = 1$$
$$C(n) = 2(2C(n-2)+2) + 2$$
$$= 2^2 C(n-2) + (2^2 + 2)$$
$$\vdots$$
$$= 2^{n-3} + \frac{2 - 2^{n-2}}{1-2} = 2^n(3/8) - 1/2 = O(2^n)$$

Exponential in the # of qubits

③

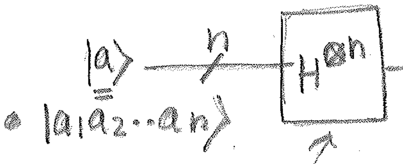


$$C = 2(n-3) + 1 = 2n - 5$$

Problem #08

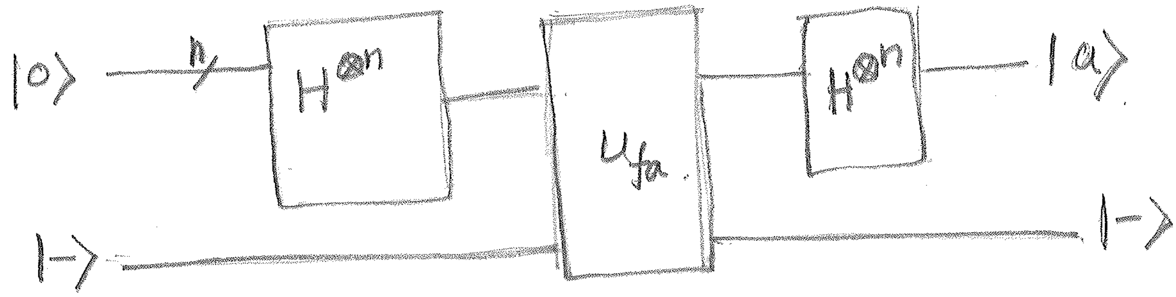
Bernstein-Vazirani Algorithm

$$f_a(x) = a\cdot x \quad , \quad a\cdot x = a_1x_1 + a_2x_2 + \dots a_nx_n.$$



$$|y\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{a\cdot x} |x\rangle$$

Product of Hadamards acting on n-qubits



$$|00\dots0\rangle|-\rangle \xrightarrow{H^{\otimes n}\otimes I} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}} |x\rangle|-\rangle$$

$$\xrightarrow{U_{f_a}} \frac{1}{\sqrt{2^{n+1}}} \sum_{x} |x\rangle \left(|f_a(x)\oplus 0\rangle - |f_a(x)\oplus 1\rangle\right)$$

If $f_a(x)=0$

$$\Rightarrow \frac{|f_a(x)\oplus 0\rangle - |f_a(x)\oplus 1\rangle}{\sqrt{2}} = |-\rangle$$

$f_a(x)=1$

$$\Rightarrow \frac{|f_a(x)\oplus 0\rangle - |f_a(x)\oplus 1\rangle}{\sqrt{2}} = -|-\rangle$$

$$\Rightarrow \frac{1}{\sqrt{2^n}} \sum_{x} |x\rangle (-1)^{f_a(x)} |-\rangle = \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{a\cdot x} |x\rangle |-\rangle$$

$$\xrightarrow{H^{\otimes n}\otimes I} |a\rangle|-\rangle$$