Exercise Sheet #05

**Problem 1** (Grover's Algorithm with multiple marked elements)

$O_f : |x\rangle \to (-1)^{f(x)}|x\rangle$. $\quad f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow O_f = \mathbb{1} - 2\sum_{x,\, f(x)=1} |x\rangle\langle x|.$$

$$|\alpha\rangle = \frac{1}{\sqrt{r}} \sum_{x,\, f(x)=1} |x\rangle \ , \quad |\beta\rangle = \frac{1}{\sqrt{N-r}} \sum_{x,\, f(x)=0} |x\rangle.$$

$$|\omega\rangle = H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{N}}\sum_x |x\rangle = \sqrt{\frac{r}{N}}|\alpha\rangle + \sqrt{\frac{N-r}{N}}|\beta\rangle.$$

Let $\sin\varphi = \sqrt{\frac{r}{N}} \Rightarrow \cos\varphi = \sqrt{\frac{N-r}{N}}$

$$\Rightarrow |\omega\rangle = \sin\varphi|\alpha\rangle + \cos\varphi|\beta\rangle$$

$$O_\omega = \mathbb{1} - 2|\omega\rangle\langle\omega|$$

**Algorithm.**

Start from $|\psi_0\rangle = |\omega\rangle$ and apply Grover's iteration $G = (-O_\omega)O_f$.

$$|\psi_{k+1}\rangle = G|\psi_k\rangle.$$

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$$O_f|\omega\rangle = \Big(1 - 2\sum_{x,\, f(x)=1}|x\rangle\langle x|\Big)\big(\sin\varphi|\alpha\rangle + \cos\varphi|\beta\rangle\big).$$

$$= -\sin\varphi|\alpha\rangle + \cos\varphi|\beta\rangle$$

$$|\psi_1\rangle = G|\psi_0\rangle = -O_\omega O_f|\omega\rangle = \big(2|\omega\rangle\langle\omega| - \mathbb{1}\big)\big(-\sin\varphi|\alpha\rangle + \cos\varphi|\beta\rangle\big)$$

$$= 2|\omega\rangle\underbrace{\left(-\sin^2\varphi + \cos^2\varphi\right)}_{\cos 2\varphi} + \big(\sin\varphi|\alpha\rangle - \cos\varphi|\beta\rangle\big).$$

$$= \left(\sin\varphi + 2\sin\varphi\cos 2\varphi\right)|\alpha\rangle + \left(-\cos\varphi + 2\cos\varphi\cos 2\varphi\right)|\beta\rangle.$$

$$= \sin 3\varphi\,|\alpha\rangle + \cos 3\varphi\,|\beta\rangle$$

$$|\psi_1\rangle = \sin\big((2(1)+1)\varphi\big)|\alpha\rangle + \cos\big((2(1)+1)\varphi\big)|\beta\rangle$$

$$\vdots$$

$$|\psi_k\rangle = \sin\big((2K+1)\varphi\big)|\alpha\rangle + \cos\big((2K+1)\varphi\big)|\beta\rangle$$

Measurement will give the result with probability 1,

if $\sin^2\big((2K+1)\varphi\big) = 1$

$$\Rightarrow \sin\big((2K+1)\varphi\big) = 1$$

$$\Rightarrow (2K+1)\varphi = \pi/2$$

$$K = \pi/4 - 1/2$$

$$\sin\varphi = \sqrt{\frac{\ell}{N}}$$

But $\ell \ll N$

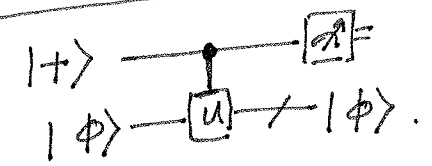
$$\Rightarrow \varphi \simeq \sqrt{\frac{\ell}{N}}$$

$$\Rightarrow K \simeq \frac{\pi}{4}\sqrt{\frac{N}{\ell}} - \frac{1}{2} \simeq \left\lceil \frac{\pi}{4}\sqrt{\frac{N}{\ell}} \right\rceil$$

Problem # 02

Phase Estimation

(1)




$$|+\rangle|\phi\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}|\phi\rangle \xrightarrow{CU} \left(\frac{|0\rangle + e^{2\pi i\phi}|1\rangle}{\sqrt{2}}\right)|\phi\rangle.$$

$$P_+ = \frac{\langle + | \left(|0\rangle + e^{2\pi i \phi}|1\rangle\right)}{2}$$

$$P_+ = \frac{|1 + e^{2\pi i \phi}|^2}{4} = \frac{2\cos 2\pi\phi + 2}{4}$$

$$P_+ = \cos^2 \pi\phi$$

$$\sqrt{N}\left|\frac{n_+}{N} - P_+\right| \xrightarrow[\text{Normality}]{\text{Asymptotic}} \mathcal{N}(0, \sigma^2).$$

$$\Rightarrow \sqrt{N}\left|\frac{n_+}{N} - P_+\right| \leq c\sigma$$

$$err = \left|\frac{n_+}{N} - P_+\right| \leq \frac{c\sigma}{\sqrt{N}}$$

$$acc = -\log(err).$$

$$\geq -\log\left(\frac{c\sigma}{\sqrt{N}}\right) = -\log(c\sigma^2) + \log(\sqrt{N}).$$

N is exponential in accuracy.

(2) (a) $CU_{n-1} \equiv \text{Cont-}U^{(2^{n-1})}$

$$|+\rangle|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\phi\rangle \xrightarrow{CU_{n-1}} \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \phi 2^{n-1}}|1\rangle\right)|\phi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.\phi_1\phi_2\cdots\phi_n)2^{n-1}}|1\rangle\right)|\phi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \phi_1\phi_2\cdots\phi_{n-1} + 2\pi i\, 0.\phi_n}|1\rangle\right)|\phi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.\phi_n}|1\rangle\right)|\phi\rangle$$

$$e^{2\pi i\, 0.\phi_n} = 1 \quad \text{when } \phi_n = 0$$

$$\quad " \quad = -1 \quad \text{when } \phi_n = 1.$$

$\Rightarrow$ measurement outcome '+' $\Rightarrow \phi_n = 0$

" '−' $\Rightarrow \phi_n = 1$

determines $\phi_n$ exactly

(b) $$CU_{n-2}|+\rangle|\phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (0.\phi_1\phi_2\cdots\phi_n)2^{n-2}}|1\rangle\right)|\phi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \phi_{n-1}} e^{2\pi i \phi_n/4}|1\rangle\right)|\phi\rangle.$$

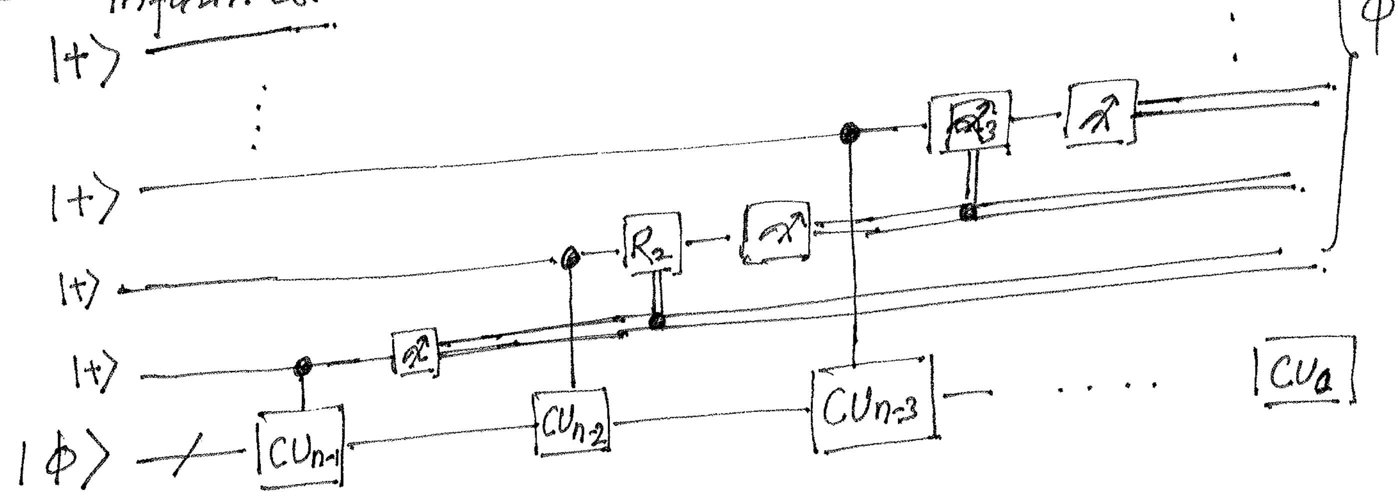
Let $R_K = \begin{pmatrix} 1 & \\ & e^{-2\pi i/2^k} \end{pmatrix}$

$$\xrightarrow{CR_2\otimes I} \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \phi_{n-1}}|1\rangle\right)|\phi\rangle$$

Measurement outcome "+" $\Rightarrow \phi_{n-1} = 0$

"−" $\Rightarrow \phi_{n-1} = 1$.

(c) Each iteration gives one bit of information




* $U^{2^k}$ must be applied $n$ times for each value of $k$ between 1 and $N$

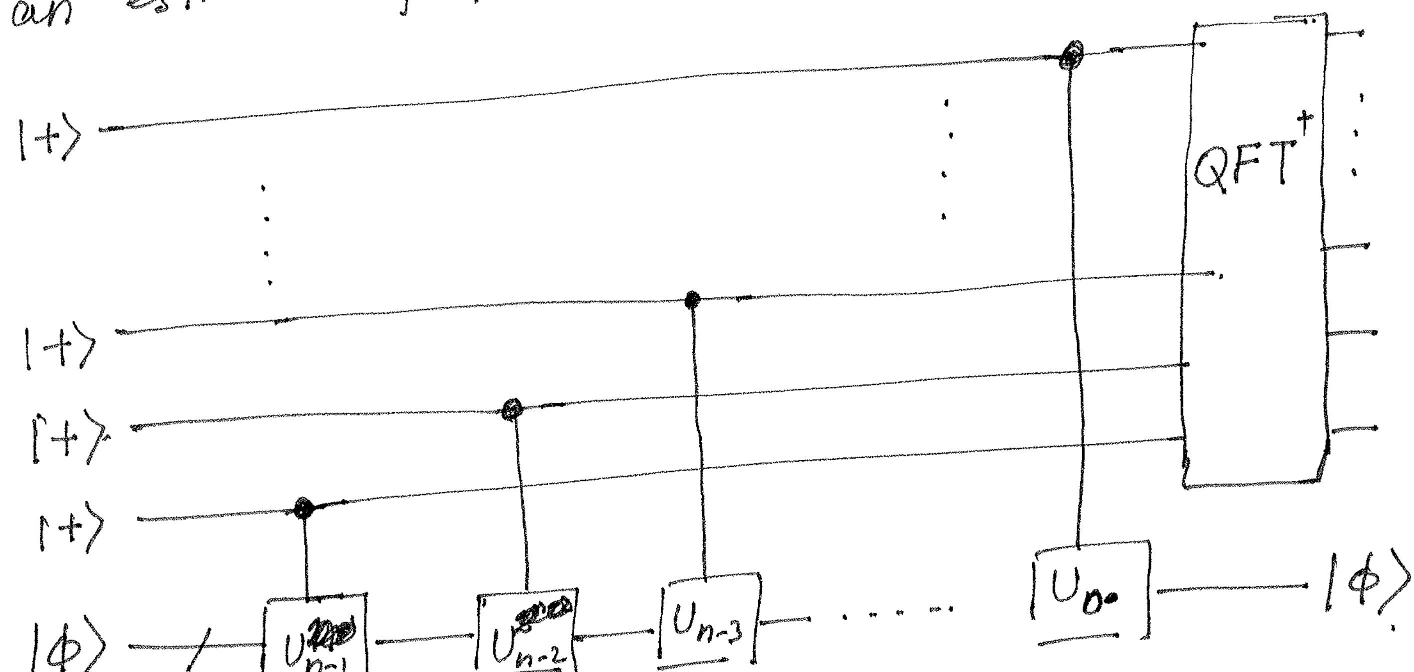
(3) $\sum_x |x\rangle|\phi\rangle \xrightarrow{U\text{-transform}} \sum_x |x\rangle U^x|\phi\rangle$

$$= \sum_x |x\rangle e^{2\pi i x\, 0.\phi_1\phi_2\ldots\phi_n}|\phi\rangle$$

$$= \sum_x e^{2\pi i x\cdot(\phi_1\phi_2\ldots\phi_n)/2^n}|x\rangle|\phi\rangle$$

$$\xrightarrow{FT_n^\dagger \otimes I} |\phi_1\phi_2\ldots\phi_n\rangle|\phi\rangle$$

Measurement on first register gives an estimate of $\phi$ with n-bits of accuracy.




$U^{2^k}$ has to be applied n-times for different values of k.

(4) Fourier transform + Measurement = One qubit gate + measurement + Classical control

(5) $|++\cdots+\rangle|\varphi\rangle \xrightarrow[\text{Estimation}]{\text{Phase}} |\varphi_1\varphi_2\cdots\varphi_n\rangle|\varphi\rangle$

$$\Rightarrow \sum_k w_k|+++\cdots+\rangle|\varphi^k\rangle \xrightarrow[\text{Estimation}]{\text{Phase}} \sum_k w_k|\varphi_1^k\varphi_2^k\cdots\varphi_n^k\rangle|\varphi^k\rangle$$

(6) $U: |x\rangle \longrightarrow |ax \pmod N\rangle$

Eigenvalues of $U = \left\{ e^{2\pi i k/r} \,\middle|\, k = 0, 1, \ldots r-1 \right\}$

$$|++\cdots+\rangle|x^k\rangle \xrightarrow[\text{Estimation}]{\text{Phase}} |x_1^k x_2^k \cdots x_n^k\rangle|x^k\rangle$$

$(x_1^k x_2^k \cdots x_n^k)$ gives a binary representation of $k/r$ with an accuracy of $n$ bits

(7) If $|x^k\rangle = \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} e(-2\pi i l k/r)|a^l \pmod N\rangle$

then $|x^k\rangle$ is an eigenvector of $U$ with eigenvalue $e(2\pi i k/r)$

$$\frac{1}{\sqrt{r}} \sum_k |x^k\rangle = \frac{1}{r} \sum_{k=0}^{r-1} \sum_{l=0}^{r-1} e\left(-2\pi i k l/r\right)|a^l \pmod N\rangle$$

$$= \frac{1}{r} \sum_{l=0}^{r-1} \underbrace{\left( \sum_{k=0}^{r-1} e\left(\left(-2\pi i l/r\right)k\right) \right)}_{r\,\delta_{0l}} |a^l \pmod N\rangle$$

$$= \frac{1}{r} \sum_{l=0}^{r-1} r\,\delta_{0l}\,|a^l \pmod N\rangle$$

$$= |1\rangle$$

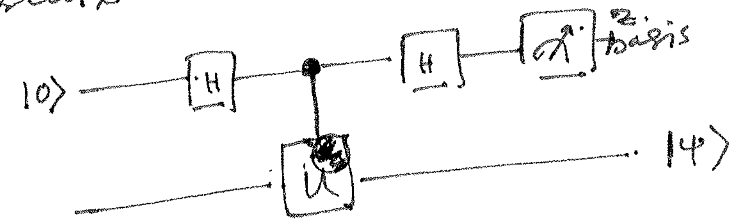
$$|++\cdots+\rangle|1\rangle \xrightarrow[\text{Estimation}]{\text{Phase}} \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} |\varphi_1^k\varphi_2^k\cdots\varphi_n^k\rangle|\varphi\rangle$$

Each measurement outcome give an estimate of $k/r$ up to $n$-bit accuracy. From $k/r$ one can obtain $r$ using continued Fraction.

# Problem #03 (Syndrome Measurement and Correction for Stabilizer Codes)

## ① 3-qubit flipcode.

(a,b) Given a unitary $U$, with eigenvalues $\{1,-1\}$ and eigenvectors $\{|U_+\rangle, |U_-\rangle\}$, one can do a measurement in $U$ basis with the following circuit




$a|U_+\rangle + b|U_-\rangle$

$$(|0\rangle + |1\rangle)(a|U_+\rangle + b|U_-\rangle) \xrightarrow{CU} |0\rangle(a|U_+\rangle + b|U_-\rangle) + |1\rangle(a|U_+\rangle - b|U_-\rangle)$$

$$\xrightarrow{H\otimes I} \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}(a|U_+\rangle + b|U_-\rangle) + \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)(a|U_+\rangle - b|U_-\rangle).$$

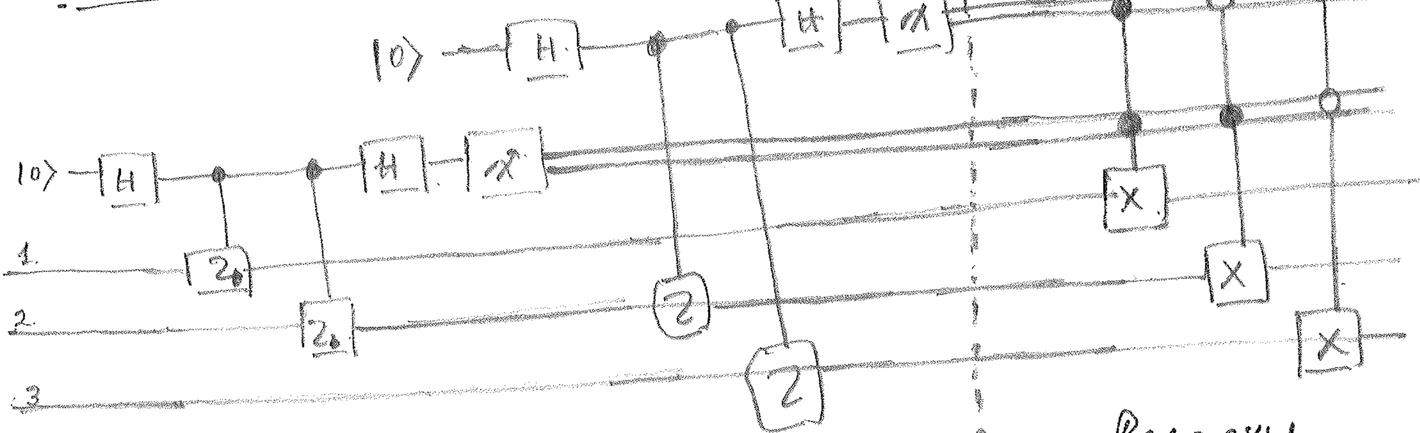
$$= a|0\rangle|U_+\rangle + b|1\rangle|U_-\rangle$$

If the outcome is

"0" $|\psi\rangle = |U_+\rangle$.

"1" $|\psi\rangle = |U_-\rangle$

Circuit a implements a POVM. $M = \{|U_+\rangle\langle U_+|, |U_-\rangle\langle U_-|\}$.




Syndrome measurement

Recovery.

| Received | Error | $S_1 = Z_1Z_2$ | $S_2 = Z_2Z_3$ | Recovery |
|---|---|---|---|---|
| $a\|000\rangle + b\|111\rangle$ | $I$ | 0 | 0 | $I$ |
| $a\|001\rangle + b\|110\rangle$ | $X_3$ | 0 | 1 | $X_3$ |
| $a\|010\rangle + b\|101\rangle$ | $X_2$ | 1 | 1 | $X_2$ |
| $a\|100\rangle + b\|011\rangle$ | $X_1$ | 1 | 0 | $X_1$ |

* Syndrom is 0 if error commutes with the Stablizer $S_i$
* Syndrom is 1 if error anticommutes with the Stablizer $S_i$

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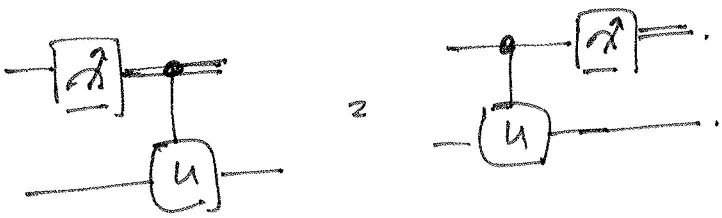
(c) This can be acheived by deferred Measurement Principle.

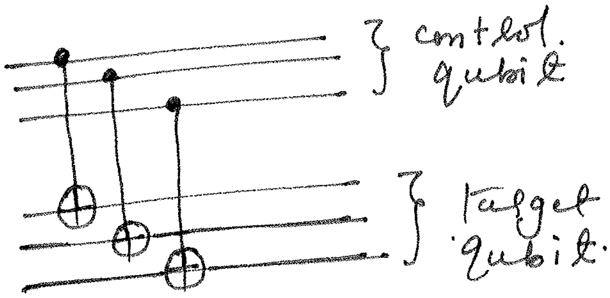



(d) $\bar{X} = XXX$

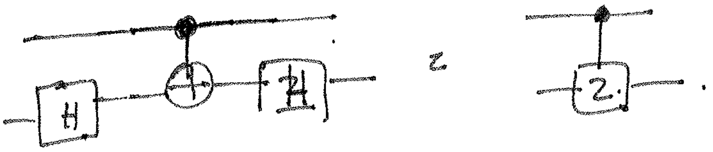
$\bar{Z} = ZZZ$

$\bar{Y} = YYY$

(e)



② Syndrom measurement and error correction can be performed by Pauli's and measurement in Z-basis.




③ Five qubit code.

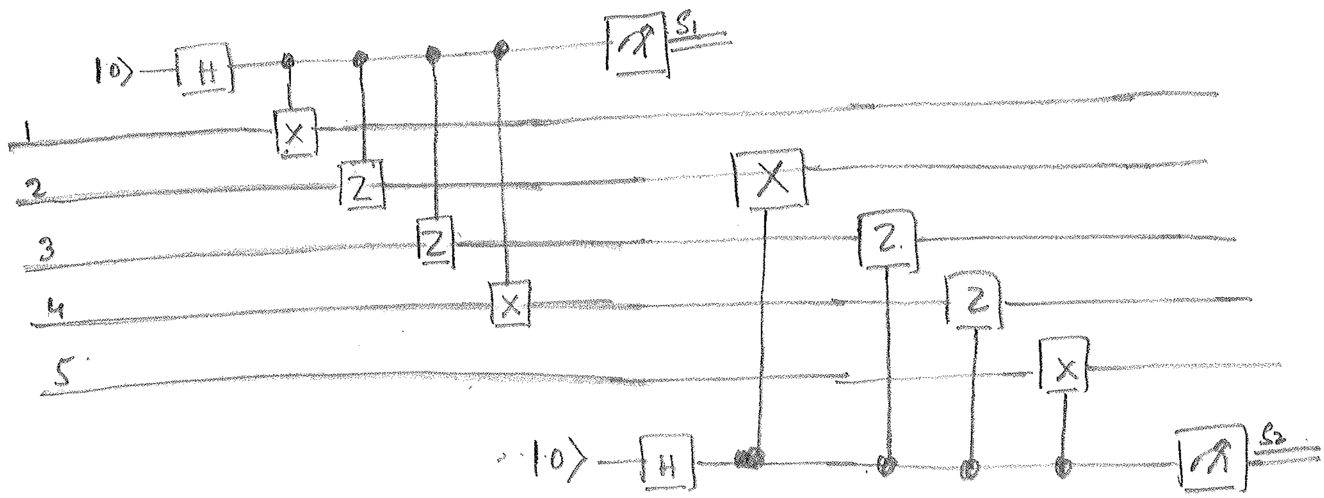
Stablizers for 5-qubit Steane code

$S_1 = XZZXI$

$S_2 = IXZZX$

$S_3 = XIXZZ$

$S_4 = ZXIXZ$.




First two syndrome measurements

(4)

| errors | $S_1 = XZZXI$ | $S_2 = IXZZX$ | $S_3 = XIXZZ$ | $S_4 = ZXIXZ$ |
|---|---|---|---|---|
| $X_1$ | 0 | 0 | 0 | 1 |
| $X_2$ | 1 | 0 | 0 | 0 |
| $X_3$ | 1 | 1 | 0 | 0 |
| $X_4$ | 0 | 1 | 1 | 0 |
| $X_5$ | 0 | 0 | 1 | 1 |
| $Y_1$ | 1 | 0 | 1 | 1 |
| $Y_2$ | 1 | 1 | 0 | 1 |
| $Y_3$ | 1 | 1 | 1 | 0 |
| $Y_4$ | 1 | 1 | 1 | 1 |
| $Y_5$ | 0 | 1 | 1 | 1 |
| $Z_1$ | 1 | 0 | 1 | 0 |
| $Z_2$ | 0 | 1 | 0 | 1 |
| $Z_3$ | 0 | 0 | 1 | 0 |
| $Z_4$ | 1 | 0 | 0 | 1 |
| $Z_5$ | 0 | 1 | 0 | 0 |
| $I$ | 0 | 0 | 0 | 0 |

Each error has unique Syndrome
the code is non-degenerate.