Lecture "Quantum Information" WS 16/17 — Exercise Sheet #2

Problem 1: Bloch sphere for mixed states.

1. Prove that any hermitian 2×2 matrix ρ with $\operatorname{tr} \rho = 1$ can be written as

$$\rho = \frac{1}{2}(\mathbb{I} + \vec{r} \cdot \vec{\sigma}) = \frac{1}{2}(\mathbb{I} + r_x X + r_y Y + r_z Z) ,$$

   where $\vec{\sigma}$ is the vector consisting of the three Pauli matrices X, Y, Z, (cf. sheet 1) and $\vec{r} \in \mathbb{R}^3$ is the *Bloch vector* of the system.

2. Show that ρ is a density operator (i.e., $\rho \geq 0$) if and only if $|\vec{r}| \leq 1$.

3. Prove that ρ is pure if and only if $|\vec{r}| = 1$.

   Note: These results show that the surface of the Bloch sphere corresponds to all pure states and its interior corresponds to all mixed states.

4. Give the Bloch vectors corresponding to the following states and draw them on the Bloch sphere:

$$\text{(a) } \rho_a = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \ ; \ \text{(b) } \rho_b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ ; \ \text{(c) } \rho_c = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} .$$

Problem 2: Measurements and filtering
Suppose that the initial state of system AB is

$$|\phi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle.$$

The goal is to obtain a maximally entangled state $|\phi_{0.5}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ with some probability after a measurement on system A.

1. Show that the operators $\Pi_0 = (|0\rangle\langle 0| + \sqrt{\gamma}|1\rangle\langle 1|)_A \otimes \mathbb{1}_B$ and $\Pi_1 = \sqrt{1-\gamma}|1\rangle\langle 1|_A \otimes \mathbb{1}_B$, with $0 \leq \gamma \leq 1$. define a POVM measurement. (Note that these describe measurements carried out on Alice's side only!)

2. Determine the outcome probabilities and the post-measurement states for each measurement outcome.

3. Find a value γ such that one of post-measurement states becomes a maximally entangled state. Calculate the corresponding probability with which the initial state becomes a maximally entangled state.

Problem 3: Quantum channels.
In this problem, we will study some commonly appearing quantum channels. In addition to the problems listed, verify for each channel that it is a CPTP map (completely positive trace preserving map) and give its Kraus representation.

1. *Dephasing channel.* This channel acts as

$$\mathcal{E}(\rho) = (1-p)\,\rho + p\,Z\rho Z\ .$$

   Show that the action of the dephasing channel on the Bloch vector is

$$(r_x, r_y, r_z) \mapsto ((1-2p)r_x, (1-2p)r_y, r_z)\ ,$$

   i.e., it preserves the component of the Bloch vector in the Z direction, while shrinking the X and Y component.

2. *Amplitude damping channel.* The amplitude damping channel is giving by the Kraus operators

$$\Pi_0 = \sqrt{\gamma}|0\rangle\langle 1|, \quad \Pi_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1| \ ,$$

where $0 \leq \gamma \leq 1$. Here, Π_0 describes a decay from $|1\rangle$ to $|0\rangle$, and γ corresponds to the decay rate.

   (a) Consider a single-qubit density operator with the following matrix representation with respect to the computation basis

   $$\rho = \begin{pmatrix} 1-p & \eta \\ \eta^* & p \end{pmatrix},$$

   where $0 \leq p \leq 1$ and η is some complex number. Find the matrix representation of this density operator after the action of the amplitude damping channel.

   (b) Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel $\mathcal{E}_1$ with parameter γ_1 and consider another amplitude damping channel $\mathcal{E}_2$ with parameter γ_2. Show that the composition of the channels, $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$, $\mathcal{E}(\rho) = \mathcal{E}_1(\mathcal{E}_2(\rho))$, is an amplitude damping channel with parameter $1 - (1-\gamma_1)(1-\gamma_2)$. Interpret this result in light of the interpretation of the γ's as a decay probability.

3. *Twirling operation.* Twirling is the process of applying a random Pauli operator (including the identity) with equal probability. Explain why this corresponds to the channel

$$\mathcal{E}(\rho) = \tfrac{1}{4}\rho + \tfrac{1}{4}X\rho X + \tfrac{1}{4}Y\rho Y + \tfrac{1}{4}Z\rho Z \ .$$

Show that the output of this channel is the maximally mixed state for any input, $\mathcal{E}(\rho) = \frac{1}{2}\mathbb{1}$.

Hint: Represent the density operator as $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$ and apply the commutation rules of the Pauli operators.

Problem 4: Purifications.

1. Consider the following three ways of expressing the maximally mixed state as an ensemble:

$$\tfrac{1}{2}\mathbb{1} = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1| = \tfrac{1}{2}|+\rangle\langle +| + \tfrac{1}{2}|-\rangle\langle -| = \tfrac{1}{4}|0\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1| + \tfrac{1}{4}|+\rangle\langle +| + \tfrac{1}{4}|-\rangle\langle -| \ .$$

   (a) Construct purifications for all three ensemble decompositions, such that the corresponding ensemble is obtained upon measuring the purifying system in the computational basis.

   (b) Show that all those purifications can be transformed into each other by acting with a unitary on the purifying systems, and explicitly construct this unitary (remember that you might have to pad with zeros).

   (c) Give POVM measurements which realize each of the three ensemble decompositions by measuring the maximally entangled state $(|00\rangle + |11\rangle)/\sqrt{2}$.

2. Consider two ensemble decompositions

$$\sum p_i|\psi_i\rangle\langle\psi_i| = \sum q_j|\phi_j\rangle\langle\phi_j| = \rho$$

of the same density matrix ρ. We will prove that in that case, the ensembles are related via

$$\sqrt{p_i}|\psi_i\rangle = \sum_j u_{ij}\sqrt{q_j}|\phi_j\rangle \qquad (1)$$

with (u_{ij}) a unitary (after possibly padding the decompositions with zeros). To start with, restrict to the case where $|\phi_i\rangle$ is an eigenbasis, and define

$$u_{ij} = \langle\phi_j|\psi_i\rangle\sqrt{p_i/q_j} \ .$$

   (a) Show that u_{ij} fulfils Eq. (1).

   (b) Show that (u_{ij}) has orthogonal columns.

   (c) Show that by padding the $|\phi_i\rangle$ decomposition with zeros, we can make (u_{ij}) unitary.

   (d) Now consider the general case, where $|\phi_i\rangle$ is not necessarily an eigenbasis.

   Hint: Connect the two ensembles by going through the eigenvalue decomposition of ρ.