Lecture "Quantum Information" WS 16/17 — Exercise Sheet #3

Problem 1: CHSH inequality I: Local hidden variable and no-signalling correlations.
Consider the scenario of the CHSH inequality. Let

$$\langle C\rangle = \langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle \ .$$

Here, $a_x = \pm 1$ and $b_y = \pm 1$ are the outcomes obtained by Alice and Bob given an input (measurement setting) of x (on Alice's side) and y (on Bob's side). The measurement is described by some joint conditional probability distribution $P(a,b|x,y)$ [i.e., $\sum_{a,b} P(a,b|x,y) = 1$],

$$\langle a_x b_y\rangle = \sum_{a,b,x,y} a\, b\, P(a,b|x,y) \ .$$

1. A local hidden variable (LHV) distribution is of the form

   $$P(a,b|x,y) = \sum_\lambda p_\lambda P_\lambda^A(a|x) P_\lambda^B(b|y) \ .$$

   Use this to derive the bound $|\langle C\rangle| \le 2$. (This should be done by explicitly using the form of $P(a,b|x,y)$, not by making any intuitive assumptions about LHV distributions. *Hint:* It can be helpful – though not necessary – to make P_λ^A and P_λ^B deterministic by introducing a new random variable λ.) Which property of $P(a,b|x,y)$ allows to obtain this bound?

2. A non-signalling is a distribution which does not allow for communication between Alice and Bob, i.e., Alice's marginal distribution $P^A(a|x) = \sum_b P(a,b|x,y)$ does not depend on Bob's input y, and vice versa. Show that no-signalling distributions can obtain the maximum possible value $|\langle C\rangle| = 4$.

3. Give a distribution $P(a,b|x,y)$ which violates no-signalling.

Problem 2: CHSH inequality II: Tsirelson's bound.
Tsirelson's inequality bounds the largest possible violation of the CHSH inequality in quantum mechanics (namely $2\sqrt{2}$). To this end, let a_0, a_1, b_0, b_1 be Hermitian operators with eigenvalues ± 1, so that

$$a_0^2 = a_1^2 = b_0^2 = b_1^2 = \mathbb{1} \ .$$

Here, a_0 and a_1 describe the two measurements of Alice, and b_0 and b_1 those of Bob; in particular, this means that Alice's and Bob's measurements commute, i.e. $[a_x, b_y] = 0$ for all $x, y = 0, 1$. Define

$$C = a_0 b_0 + a_1 b_0 + a_0 b_1 - a_1 b_1 \ .$$

1. Determine C^2.

2. The norm of a bounded operator M is defined by

   $$\|M\| = \sup_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} \ ,$$

   that is, the norm of M is the maximum eigenvalue of $\sqrt{M^\dagger M}$. Verify that the norm has the properties

   $$\|MN\| \le \|M\|\|N\| \ ,$$
   $$\|M+N\| \le \|M\| + \|N\| \ .$$

3. Find an upper bound on the norm $\|C^2\|$.

4. Show that for Hermitian operators $\|C^2\| = \|C\|^2$. Use this to obtain an upper bound on $\|C\|$.

5. Explain how this inequality gives a bound on the maximum possible violation of the CHSH inequality in quantum mechanics. This is known as Tsirelson's bound, or Tsirelson's inequality.

Problem 3: Remote state preparation.
Remote state preparation is a variation on the teleportation protocol. We consider a simple example of a remote state preparation protocol. Suppose Alice possesses a classical description of a state $|\psi\rangle = (|0\rangle + e^{i\phi}|1\rangle)/\sqrt{2}$ (on the equator of the Bloch sphere) and she shares an ebit $|\Phi^+\rangle$ with Bob. Alice would like to prepare this state on Bob's system. Show that Alice can prepare this state on Bob's system if she measures her system A in the $(\psi, \psi^\perp)$ basis, transmits one classical bit, and Bob performs a recovery operation conditional on the classical information, but independent of state $|\psi\rangle$.

Problem 4: LOCC protocols.
Suppose $|\psi\rangle$ can be transformed to $|\phi\rangle$ by LOCC. A general LOCC protocol can involve an arbitrary number of rounds of measurement and classical communication. In this problem, we will show that any LOCC protocol can be realized in a single round with only one-way communication, i.e., a protocol involving just the following steps: Alice performs a single measurement described by measurement operators K_j, sends the result j to Bob, and Bob performs a unitary operation U_j on his system.
The idea is to show that the effect of any measurement which Bob can do can be simulated by Alice (with one small caveat) so all Bob's actions can actually be replaced by actions by Alice.

1. First, suppose that Bob performs a measurement with operators $M_j = \sum_{kl} M_{j,kl}|k\rangle_B\langle l|_B$ on a pure state $|\psi\rangle_{AB} = \sum \lambda_l |l\rangle_A|l\rangle_B$, with resulting state denoted as $|\psi_j\rangle$. Now suppose that Alice performs a measurement with operators $N_j = \sum_{kl} M_{j,kl}|k\rangle_A\langle l|_A$ on a pure state $|\psi\rangle$, with resulting state denoted as $|\phi_j\rangle$. Show that there exist unitaries U_j on system A and V_j on system B such that $|\psi_j\rangle = (U_j \otimes V_j)|\phi_j\rangle$.
2. Use this to explain how any multi-round protocol can be implemented with one measurement done by Alice followed by a unitary operation done by Bob which depends on Alice's outcome.

Problem 5: Majorization.

1. Show that majorization defines a pre-order on $\mathbb{R}^d_{\geq 0}$, i.e.,
   (a) $x \prec x$ (reflexivity)
   (b) $x \prec y$ and $y \prec z \;\Rightarrow\; x = z$ (transitivity)

   Show that if we restrict to the *ordered* vectors $x = (x_1, \dots, x_d)$, $x_1 \geq x_2 \geq \cdots \geq 0$, it defines a partial order, i.e., we additionally have that

   (c) $x \prec y$ and $y \prec x \;\Rightarrow\; x = y$ (antisymmetry)
2. Show that the order given by majorization is not total, i.e., find two vectors x and y which are not related by majorization.
3. Let f be a convex function, and define $F(x) = \sum f(x)$. Show that $x \prec y$ implies that $F(x) \leq F(y)$, a property known as "Schur convexity". (*Hint*: Use the result given in the lecture which relates majorization $x \prec y$ to the possibility to obtain x from y by a random process.)
4. Use the previous problem to show that the entanglement $E(|\psi\rangle) = S(\mathrm{tr}_A|\psi\rangle\langle\psi|)$ cannot be increased by LOCC. (*Hint:* $x\log(x)$ is convex.)
5. Show that $x \prec y$ if and only if for all real t, $\sum_{j=1}^d \max\{x_j - t, 0\} \leq \sum_{j=1}^d \max\{y_j - t, 0\}$, and $\sum_{j=1}^d x_j = \sum_{j=1}^d y_j$.
6. Use the previous result (point 5.) to show that the set of x such that $x \prec y$ is convex.