Lecture “Quantum Information” WS 16/17 — Exercise Sheet #4

Problem 1: Decay of entanglement.
Consider a Bell state $\rho = |\Phi^+\rangle\langle\Phi^+|$, where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Superposition states like ρ generally are not stable, but decay over time. A typical evolution is that the populations (i.e., the diagonal elements) becom equal, while the off-diagonal elements decay to zero. Suppose that the state evolves as

$$\rho(t) = p_+|00\rangle\langle 00| + p_-|01\rangle\langle 01| + p_-|10\rangle\langle 10| + p_+|11\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|00\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|11\rangle\langle 00| ,$$

with $p_\pm = \frac{1}{4}(1 \pm e^{-t/T_1})$.
For sufficiently long times, this state tends to $\lim_{t\to\infty} \rho(t) = \frac{1}{4}\mathbb{I}$, the maximally mixed state.

1. Write a matrix form of state $\rho(t)$.
2. Take its partial transpose $\rho(t)^{T_B}$ and write its matrix form.
3. Calculate the eigenvalues of $\rho(t)^{T_B}$. (You may use a computer algebra system, though it should not be necessary.)
4. Sketch how the eigenvalues change over time for $T_1 = T_2 = 1$. What it the asymptotic limit? Also, compute and plot the negativity $\mathcal{N}(\rho(t))$ and log-negativity $E_N(\rho(t))$ as a function of time.
5. Is the state $\rho(t = 0)$ is entangled or separable? Find time after which state $\rho(t)$ becomes separable.

Problem 2: Bell inequalities and witnesses.
The CHSH operator

$$C = \vec{n}_1\vec{\sigma} \otimes \vec{n}_0\vec{\sigma} + \vec{n}_1\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} + \vec{n}_3\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} - \vec{n}_3\vec{\sigma} \otimes \vec{n}_0\vec{\sigma}$$

with $\vec{n}_k = (\cos(k\pi/4), 0, \sin(k\pi/4))$ has the property that $|\mathrm{tr}[C\rho]| \le 2$ for all ρ which describe a local hidden variable (LHV) model. Note that any separable state $\rho = \sum p_i\rho_i^A \otimes \rho_i^B$ describes a LHV model.

1. Use C to construct an entanglement witness W. Provide an explicit form of the witness. (You may use that all separable states describe LHV models to prove that $\mathrm{tr}[W\rho] \ge 0$.)
2. In which range of λ does this witness detect Werner states $\rho(\lambda) = \lambda|\Psi^-\rangle\langle\Psi^-| + \frac{1-\lambda}{4}\mathbb{I}$, with $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$? How does it compare to the entanglement witness $W = \mathbb{F}$ discussed in the lecture?

Problem 3: Witnesses and reduction criterion.
Consider $W = \mathbb{I} - d|\Omega\rangle\langle\Omega|$, with $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i, i\rangle$.

1. Show that $\mathrm{tr}[W\rho] \ge 0$ for separable states ρ, i.e., W is an entanglement witness.
2. Consider the family
$$\rho_{\mathrm{iso}}(\lambda) = \lambda\frac{\mathbb{I}}{d^2} + (1-\lambda)|\Omega\rangle\langle\Omega|$$
of *isotropic states*. In which range of λ is $\rho_{\mathrm{iso}}(\lambda) \ge 0$? In which range of λ does W detect that $\rho_{\mathrm{iso}}(\lambda)$ is entangled?
3. Consider the case $d = 2$. What does W do on the antisymmetric state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$?
4. Derive the positive map Λ corresponding to the witness W. Prove directly that it is indeed a positive map.
5. In which range of λ does Λ detect that $\rho_{\mathrm{iso}}(\lambda)$ is entangled? What does Λ do on the antisymmetric state?

Problem 4: One-qubit unitaries.

1. Show that for any U such that $U^2 = I$ the following holds $\exp\{i\phi U\} = \cos\phi\, I - i \sin\phi\, U$.

2. Verify that $R_z(\phi)$ is indeed rotates a vector $\hat{r} = (r_x, r_y, r_z)$ around z-axis by angle ϕ, i.e. let ρ has a Bloch vector $\hat{r}$, find Bloch vector of a rotated state $\rho' = R_z(\phi)\rho R_z(\phi)^\dagger$.

3. Show that up to a global phase any unitary one-qubit transformation U can be implemented with three rotations about x and z-axes, i.e. find angles α, β, γ and α', β', γ' such that $U = R_x(\alpha)R_z(\beta)R_x(\gamma)$ and $U = R_z(\alpha')R_x(\beta')R_z(\gamma')$. (*Hint:* Up to a global phase factor any unitary transformation on a single qubit is a rotation $U = R_{\hat{n}}(\phi)$ by an angle ϕ about axis $\hat{n} = (n_x, n_y, n_z)$.)

(*Note:* There is nothing specific about the choice of x and z axes, one may e.g. choose y and z instead, i.e. for some angles α, β, γ the following holds $U = R_z(\alpha)R_y(\beta)R_z(\gamma)$.)

Problem 5: Controlled-U gate.

In the following, we will show that for any unitary matrix U controlled-U gate can be realized using only one-qubit and CNOT gates.

1. Use previous exercise to show that for a special unitary matrix $U \in SU(2)$ (i.e. $\det(U) = 1$) there exist matrices $A, B, C \in SU(2)$ such that $ABC = I$ and $AXBXC = U$, where X is one of the Pauli matrices.

2. Based on this, find a realization of controlled-U gate (for any unitary U) that uses only the matrices A, B, and C, CNOT gates, and an additional one-qubit gate E that is used to adjust the global phase.

Problem 6: Gate identities.
Verify the following gate identities given in the lecture:

1. Verify the identities for the behavior of the CNOT gate when conjugating it with Hadamard gates.

2. Verify the construction for the Toffoli gate using controlled-V gates.

(*Note:* While both of these identities can be verified by multiplying out the matrices, it is more instructive to treat the control qubits as "classical", i.e., consider each of their values in the computational basis.)

Problem 7: n-qubit Toffoli gate.
An n-qubit Toffoli gate is a Toffoli gate with $n-1$ controls; i.e., it flips the n'th bit if and only if the other $n-1$ bits are all one.

1. Show that the n-qubit Toffoli gate can be implemented using two $n-1$-qubit Toffoli gates and two regular 3-qubit Toffoli gates using one ancillary qubit.

2. Decomposing every gate into 3-qubit Toffoli gates, how many 3-qubit Toffoli gates do you need to construct the n-qubit Toffoli gate?

3. Find a construction which is more efficient in terms of the scaling of the number 3-qubit Toffoli gates used, at the cost of using more ancillas. (A linear number of 3-fold Toffoli gates should suffice.)

(*Hint:* Remember that the Toffoli gate can be used to build a logical AND gate using ancillas.)

Problem 8: The Bernstein-Vazirani algorithm.
This is a variation of the Deutsch-Jozsa problem. Suppose that the quantum black box computes one of the functions f_a, where $f_a(x) = a \cdot x$ and a is an n-bit string. The task is to determinate a. Show that Deutsch-Jozsa algorithm can solve this problem, i.e. can find the n-bit string a with probability one.