We have

$$\sum \lambda_i |\phi_i^A\rangle\langle\phi_i^A| = \mathrm{tr}_B |\phi\rangle\langle\phi| = \mathrm{tr}_B |\psi\rangle\langle\psi| = \sum \mu_i |\psi_i^A\rangle\langle\psi_i^A|$$

$\Rightarrow \lambda_i = \mu_i$, $|\phi_i^A\rangle = |\psi_i^A\rangle$ (up to phase)

if λ_i non-degen. (degen. $\rightarrow$ HW)

Now choose $\mathcal{U}$ s.th. $\mathcal{U}|\phi_i^B\rangle = |\psi_i^B\rangle$ $\forall i$ ($\Rightarrow \mathcal{U}$ unitary)

$\Rightarrow$ $|\psi\rangle = (\mathbb{1} \otimes \mathcal{U})|\phi\rangle$.

All purifications are related by a unitary on the purifying system.

(Note: Closely related to unitary equivalence of ensemble decompositions $\rightarrow$ HW!)

36. Mixed states – unitary evolution + projective measurement

Unitary evolution of mixed state

How does a mixed state ρ_A evolve under a unitary $\mathcal{U}_A$?

Consider purification $|\psi\rangle_{AB}$, $\mathrm{tr}_B|\psi\rangle\langle\psi| = \rho_A$.

$$|\psi\rangle \longmapsto (\mathcal{U}_A \otimes \mathbb{1}_B)|\psi\rangle$$

$$\Rightarrow \rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi| \longmapsto \mathrm{tr}_B\left[(U_A \otimes \mathbb{1}_B)|\psi\rangle\langle\psi|(U_A^\dagger \otimes \mathbb{1}_B)\right]$$

$$= U_A \cdot \mathrm{tr}_B\left[(\mathbb{1}_A \otimes \mathbb{1}_B)|\psi\rangle\langle\psi|(\mathbb{1}_A \otimes \mathbb{1}_B)\right] U_A^\dagger$$

$$= \underline{\underline{U_A \rho_A U_A^\dagger}}$$

(Alt. derivation: $\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i|$ & $|\psi_i\rangle \mapsto U_A|\psi_i\rangle$)

<u>Measurement of mixed states:</u>

Proj. measurement E_n:

Have seen: $p_n = \mathrm{tr}[E_n \rho_A]$.

Post-meas. state:

$$\rho_{A,n} = \frac{1}{p_n} \mathrm{tr}_B\left[(E_n \otimes \mathbb{1})|\psi\rangle\langle\psi|(E_n^\dagger \otimes \mathbb{1})\right]$$

$$= E_n \rho_A E_n^\dagger .$$

5. POVM measurements

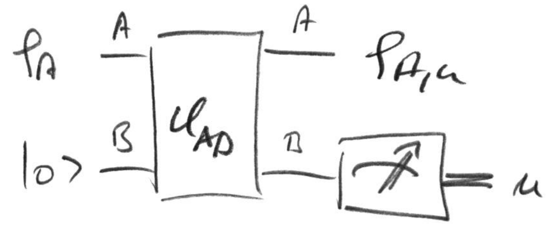
Have seen: add'l system B → more rich situation

What measurements can we realize by adding extra system?

Idea: i) Add "ancilla" B in state $|0\rangle$

ii) act w/ unitary U_{AB}

iii) measure B in $|0\rangle, \dots, |d_B - 1\rangle$



Post-meas. state (un-normalized):

$$\tilde{\rho}_u^A = \langle u|_B \, U \left(\rho_A \otimes |0\rangle\langle 0|_B \right) U^\dagger |u\rangle_B$$

$$= M_u \rho_A M_u^\dagger \,, \quad \text{with } M_u := \langle u|_B \, U |0\rangle_B$$

$$\equiv \left(\mathbb{1}_A \otimes \langle u|_B \right) U \left(\mathbb{1}_A \otimes |0\rangle_B \right)$$

and $\underline{\underline{p_u}} = \mathrm{tr}\, \tilde{\rho}_u^A = \mathrm{tr}\left(M_u \rho_A M_u^\dagger \right) = \underline{\mathrm{tr}\left(M_u^\dagger M_u \rho_A \right)}$.

We have

$$\underline{\underline{\sum M_u^\dagger M_u}} = \sum_u \langle 0|_B U^\dagger |u\rangle_B \langle u|_B U |0\rangle_B = \langle 0|_B \mathbb{1}_{AB} |0\rangle_B = \underline{\underline{\mathbb{1}_A}}$$

(Ensures $\sum p_m = \sum \mathrm{tr}(M_m^\dagger M_m \rho_A) = \mathrm{tr}(\rho_A) = 1$)

Definition: A set $\{F_m = M_m^\dagger M_m\}$ with $0 \le F_m \le \mathbb{1}$, $\sum F_m = \mathbb{1}$, is called <u>positive operator-valued measure</u>, and the corresp. measurement w/ outcome probs $p_m = \mathrm{tr}[M_m^\dagger M_m \rho] = \mathrm{tr}[F_m \rho]$ a <u>POVM measurement</u>.

(<u>Note</u>: By "fine-graining" outcomes — $F_m = \sum_{n_m \in \mathrm{fg}(m)} M_{n_m}^\dagger M_{n_m}$ — we can realize <u>any</u> $\{F_m\}$ s.th. $0 \le F_m \le \mathbb{1}$ & $\sum F_m = \mathbb{1}$
$\Rightarrow$ most general measurement!)

Can any $\{M_m\}$ w/ $\sum M_m^\dagger M_m = \mathbb{1}$ be realized by extensions + unitaries?

$$\begin{pmatrix} M_0 \\ \vdots \\ M_{d-1} \end{pmatrix} \quad \begin{array}{c} \sum M_m^\dagger M_m = \mathbb{1} : \text{orth. columns} \\ \Longrightarrow \\ \text{can be extended to unitary} \end{array} \qquad U = \begin{array}{c} \\ \langle 0|_B \\ \langle 1|_B \\ \vdots \\ \langle d-1|_B \end{array} \begin{array}{c} \begin{array}{ccc} |0\rangle_B & |1\rangle_B & \dots \end{array} \\ \begin{pmatrix} M_0 & \cdot & \cdot \\ M_1 & \cdot & \cdot \\ \vdots & & \\ M_{d-1} & \cdot & \cdot \end{pmatrix} \end{array}$$

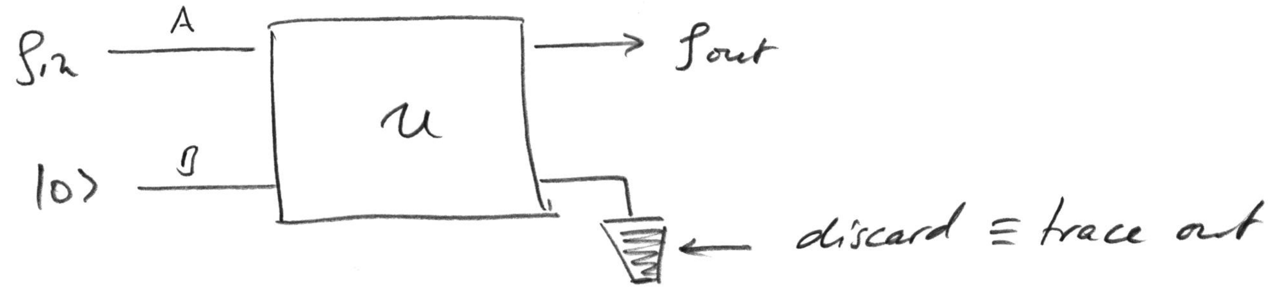
i.e.: $\langle m|_B U |0\rangle_B = M_m$.

$\Rightarrow$ any POVM meas. $\{M_m\}$ can be realized by unitary U + projective measurement!

6. General evolution – superoperators

Q.: What is the most general physical map on density matrices ("superoperator")?

Idea: Try to add ancilla:



$$\rho \mapsto \mathcal{E}(\rho) = \mathrm{tr}_B\left[U(\rho\otimes|0\rangle\langle 0|)U^\dagger\right]$$

$$= \sum_n \underbrace{\langle n|_B U|0\rangle_B}_{=:\,M_n} \rho \langle 0|_B U^\dagger |n\rangle_B$$

$$= \sum_n M_n \rho M_n^\dagger .$$

(Note: trace in diff. basis $\Rightarrow$ diff. M_n: not unique!)

Properties of M_n? As before:

$$\sum M_n^\dagger M_n = \sum \langle 0|_B U|n\rangle_B \langle n|_B U^\dagger|0\rangle_B = \mathbb{1}_A .$$

Kraus representation:

We call

$$\mathcal{E}(\rho) = \sum M_n \rho M_n^{+} ; \quad \sum M_n^{+} M_n = \mathbb{1}$$

the <u>Kraus form</u> or <u>Kraus representation</u> of $\mathcal{E}$.

<u>Note</u>: Any such $\mathcal{E}$ can be realized by ancilla + unitary (cf. POVM). In fact, $\mathcal{E}$ can be seen as POVM where we ignore result (meas. by environment).

Is this the most general physical evolution?

Conditions for physical evolution $\mathcal{E}$:

(i) hermiticity-preserving: $\rho = \rho^{+} \Rightarrow \mathcal{E}(\rho) = \mathcal{E}(\rho)^{+}$

(ii) positive: $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) \geq 0$.

(iii) trace-preserving: $\mathrm{tr}(\rho) = 1 \Rightarrow \mathrm{tr}(\mathcal{E}(\rho)) = 1$

(iv) linear: $\mathcal{E}(\rho + \lambda \sigma) = \mathcal{E}(\rho) + \lambda \mathcal{E}(\sigma)$

(Note: w/out linearity, ensemble interpretation breaks down $\rightarrow$ HW)

Is this sufficient? <u>NO!</u>

→ $\mathcal{E}$ should still be a physical map when it acts on <u>part of a larger system</u>.

(v) <u>complete positivity</u>:

$$\rho_{AB} \geq 0 \Rightarrow (\mathcal{E}_A \otimes \mathbb{1}_B)(\rho_{AB}) \geq 0$$

(<u>Note</u>: $\mathcal{E}_A \otimes \mathbb{1}_B$ def. on basis: $(\mathcal{E}_A \otimes \mathbb{1}_B)(M \otimes N) = \mathcal{E}_A(M) \otimes N$ + linearity)

We call $\mathcal{E}$ satisfying (i) – (v) a <u>completely positive trace preserving (CPTP) map</u>, or a <u>quantum channel</u>.

Are there maps which are <u>positive</u> ((i)-(iv)) but <u>not CP</u>?

<u>YES</u>: E.g. "transposition channel"

$$\mathcal{E}(\rho) = \rho^T$$

$$(\mathcal{E} \otimes \mathbb{1})(\rho_{AB}) = \rho_{AB}^{T_A} \quad \text{"partial transpose"}$$

E.g.: $|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$$(\mathcal{E}\otimes\mathbb{1})(|\Omega\rangle\langle\Omega|) = (|\Omega\rangle\langle\Omega|)^{T_B} = \frac{1}{2}\Big[\underbrace{|00\rangle\langle 00|}_{T_B} + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\Big]$$

$$= \frac{1}{2}\begin{pmatrix} 1 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 1 \end{pmatrix} \not\geq 0 \quad !$$

Note: Positive but not CP maps can serve as <u>entanglement witnesses</u>: $(\mathcal{E}\otimes\mathbb{1})(\rho) \geq 0$ for all unentangled states, so $(\mathcal{E}\otimes\mathbb{1})(\sigma) \not\geq 0 \Rightarrow \sigma$ entangled.

$\mathcal{E}$ is in Kraus form $\Rightarrow$ $\mathcal{E}$ CPTP

(by construction or by explicit inspection of

$$(\mathcal{E}\otimes\mathbb{1})(\rho) = \sum \underbrace{(M_n\otimes\mathbb{1})\rho(M_n\otimes\mathbb{1})^\dagger}_{\geq 0} \quad)$$

Are also all CPTP maps of Kraus form?

Choi-Jamiołkowski isomorphism

Let $\mathcal{C} = \{ \mathcal{E} \mid \mathcal{E} \text{ CPTP} \}$ the space of CPTP maps on the density operators on $\mathbb{C}^d$, and

$$\mathcal{S} := \{ \sigma_{AB} \in \underbrace{\mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)}_{\text{linear operators on } \mathbb{C}^d \otimes \mathbb{C}^d} \mid \sigma_{AB} \geq 0,\ \mathrm{tr}_A(\sigma_{AB}) = \tfrac{1}{d} \mathbb{1} \}.$$

Then, the map

$$\hat{X} : \mathcal{E} \longmapsto \sigma_{AB} = (\mathcal{E}_A \otimes \mathbb{1}_B)(|\Omega\rangle\langle\Omega|), \quad |\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i,i\rangle$$

defines an isomorphism between $\mathcal{C}$ and $\mathcal{S}$, the <u>Choi-Jamiołkowski isomorphism</u>, with σ_{AB} the <u>Choi state</u> of $\mathcal{E}$. The inverse map is given by

$$\hat{Y} : \sigma_{AB} \longmapsto \mathcal{F}; \text{ with } \mathcal{F}(\rho) = d \cdot \mathrm{tr}_B [\sigma_{AB} \cdot (\mathbb{1} \otimes \rho^T)]$$

<u>Proof</u>: We need to show

(i) $\hat{Y}\hat{X} = \mathbb{1}_{\mathcal{C}}$

(ii) $\hat{X}\hat{Y} = \mathbb{1}_{\mathcal{S}}$

(iii) $\mathrm{Im}\, \hat{X} = \{ \hat{X}(\mathcal{E}) \mid \mathcal{E} \in \mathcal{C} \} \subset \mathcal{S}$

(iv) $\mathrm{Im}\, \hat{Y} \subset \mathcal{C}$.

(i) $\hat{Y}(\underbrace{\hat{X}(\mathcal{E})}_{\equiv \sigma_{AB}})(\rho) = d \cdot tr_B[\underbrace{\hat{X}(\mathcal{E})}_{\equiv \sigma_{AB}} \cdot (\mathbb{1}_A \otimes \rho^T)]$

$$= tr_B\left[\left((\mathcal{E} \otimes \mathbb{1}_B)(|\Omega\rangle\langle\Omega|)\right)(\mathbb{1}_A \otimes \rho^T)\right]$$

$$= \sum_{ij} tr_B\left[\left((\mathcal{E} \otimes \mathbb{1}_B)(|i\rangle\langle j| \otimes |i\rangle\langle j|)\right) \cdot (\mathbb{1}_A \otimes \rho^T)\right]$$

$$= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \cdot \underbrace{tr_B[|i\rangle\langle j| \rho^T]}_{= \langle j|\rho^T|i\rangle = \rho_{ij}}$$

$$= \mathcal{E}\left(\sum \rho_{ij} |i\rangle\langle j|\right) = \underline{\underline{\mathcal{E}(\rho)}}.$$

✓

(ii) $\hat{X}(\underbrace{\hat{Y}(\sigma_{AB})}_{\equiv \mathcal{F}}) = (\hat{Y}(\sigma_{AB}) \otimes \mathbb{1})(|\Omega\rangle\langle\Omega|)$

$$= \frac{1}{d} \sum_{ij} \underbrace{(\hat{Y}(\sigma_{AB}))}_{\equiv \mathcal{F}}(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

$$= \frac{1}{\cancel{d}} \sum_{ij} \cancel{d} \cdot tr_B[\sigma_{AB} \cdot (\mathbb{1} \otimes (|i\rangle\langle j|)^T)] \otimes |i\rangle\langle j|$$

$$= \sum_{ij} \langle i|_B \sigma_{AB} |j\rangle_B \otimes |i\rangle_B \langle j|_B = \underline{\underline{\sigma_{AB}}}.$$

✓

(iii) $\sigma_{AB} = \hat{X}(\mathcal{E}) \geq 0$ by construction ($\mathcal{E}$ CPTP).

$$\operatorname{tr}_A(\sigma_{AB}) = \frac{1}{d}\sum_{ij} \underbrace{\operatorname{tr}_A\left[\mathcal{E}(|i\rangle\langle j|)\right.}_{\operatorname{tr}\mathcal{E}(|i\rangle\langle j|) = \operatorname{tr}(|i\rangle\langle j|) = \delta_{ij}} \left.\otimes |i\rangle\langle j|\right] = \frac{1}{d}\mathbb{1}_B .$$

$$\Rightarrow \sigma_{AB} = \hat{X}(\mathcal{E}) \in \mathcal{S} \text{ for } \mathcal{E} \in \mathcal{C}.$$

✓

(iv) Let $\sigma_{AB} \in \mathcal{S}$. Write $\sigma_{AB} = \sum_k |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|$

unnormalized.

We have $\underline{|\tilde{\psi}_k\rangle} = \sum_{ij} m_k^{ij} |j\rangle|i\rangle = \frac{1}{\sqrt{d}}\sum_i (\Pi_k \otimes \mathbb{1})|i\rangle|i\rangle$.

$$= \underline{(\Pi_k \otimes \mathbb{1})|\Omega\rangle}$$

$$\Rightarrow \sigma_{AB} = \sum_k (\Pi_k \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(\Pi_k \otimes \mathbb{1})^\dagger, \text{ and}$$

$$\underline{\left(\hat{Y}(\sigma_{AB})\right)(\rho)} = d \operatorname{tr}_B\left[\sigma_{AB}\cdot(\mathbb{1}\otimes\rho^T)\right]$$

$$= d \operatorname{tr}_B\left[\left(\sum_k (\Pi_k \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(\Pi_k\otimes\mathbb{1})^\dagger\right)(\mathbb{1}\otimes\rho^T)\right]$$

$$= d \sum_k \Pi_k \underbrace{\operatorname{tr}_B\left[|\Omega\rangle\langle\Omega|\cdot(\mathbb{1}\otimes\rho^T)\right]}_{=\frac{1}{d}\sum_{ij}|i\rangle\langle j|_A \operatorname{tr}_B\left[|i\rangle\langle j|\rho^T\right] = \frac{1}{d}\rho} \Pi_k^\dagger$$

$$= \underline{\sum_k \Pi_k \rho \Pi_k^\dagger}.$$

and $\frac{1}{d}\mathbb{1} = \mathrm{tr}_A\,\sigma_{AB} = \mathrm{tr}_A\Big[\sum_k (\Pi_k \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(\Pi_k \otimes \mathbb{1})^\dagger\Big]$

$$= \sum_k \mathrm{tr}_A\Big[\big(\Pi_k^\dagger \Pi_k \otimes \mathbb{1}\big)|\Omega\rangle\langle\Omega|\Big]$$

$$= \frac{1}{d}\sum_{ijk} \mathrm{tr}\big(\Pi_k^\dagger \Pi_k\, |i\rangle\langle j|\big) \otimes |i\rangle\langle j|$$

$\Rightarrow \quad \langle j|\Pi_k^\dagger \Pi_k|i\rangle = \mathrm{tr}\big(\Pi_k^\dagger \Pi_k\, |i\rangle\langle j|\big) = \delta_{ij}$

$\Rightarrow \quad \sum_k \Pi_k^\dagger \Pi_k = \mathbb{1}.$

$\Rightarrow \quad \hat{Y}(\sigma_{AB}) \in \mathcal{C}$ for $\sigma_{AB} \in \mathcal{S}$. ✓ $\boxtimes$

Note: The isomorphism still holds if we drop trace preserving and $\mathrm{tr}_A\,\sigma_{AB} = \frac{1}{d}\mathbb{1}$, respectively.

__Corollary (from (iv)):__ All CPTP maps are of Kraus form (and can thus be realized w/ ancilla + unitary + tracing).

7. Axioms ("mixed version")

- States are linear operators with

$$\rho \geq 0$$
$$\mathrm{tr}\,\rho = 1.$$

- Evolutions are completely positive trace preserving (CPTP) maps

$$\mathcal{E}(\rho) = \sum M_k \rho M_k^\dagger \quad \text{with } \sum M_k^\dagger M_k = \mathbb{1}.$$

- Measurements act as

$$\rho \mapsto \rho_k = \frac{M_k \rho M_k^\dagger}{\mathrm{tr}(M_k \rho M_k^\dagger)},$$

with prob. $p_k = \mathrm{tr}(M_k^\dagger M_k \rho)$ and $\sum M_k^\dagger M_k = \mathbb{1}$.