IV. Entanglement

1. Introduction

Consider a <u>bipartite pure state</u> $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

$|\psi\rangle$ is a <u>product state</u> if it can be written as

$$|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle.$$

If $|\psi\rangle$ <u>cannot</u> be written in this form, we say <u>$|\psi\rangle$ is entangled.</u>

<u>Characterization:</u>

- <u>Schmidt coefficients</u> $\vec{\lambda} = (\lambda_1, \lambda_2, \dots)$

  <u>product state:</u> $\vec{\lambda} = (1, 0, \dots 0)$

  <u>entangled state:</u> $\vec{\lambda} = (\lambda_1, \lambda_2 \neq 0, \dots)$

- <u>Reduced density matrix:</u>

  <u>product state:</u> $\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi| = |\phi_A\rangle\langle\phi_A|$

  $\rho_B = |\phi_B\rangle\langle\phi_B|.$

  $\Rightarrow$ reduced state is pure.

And conversely:

ρ_A pure $\Rightarrow$ Schmidt coeffs $(1,0,\dots 0)$

$\Rightarrow |\psi\rangle = |\chi_A\rangle \otimes |\chi_B\rangle$ & $|\chi_A\rangle, |\chi_B\rangle$ determined by ρ_A, ρ_B.

Measured by purity: $\operatorname{tr} \rho_A^2 = 1$ for pure ρ_A's or some entropies of Schmidt coeffs.

entangled state:

ρ_A mixed $\Rightarrow \operatorname{tr} \rho_A^2 < 1$.

Entangled states "different":

- Cannot describe parts independently
- meas. outcomes correlated
- will see: suitable for non-trivial tasks.

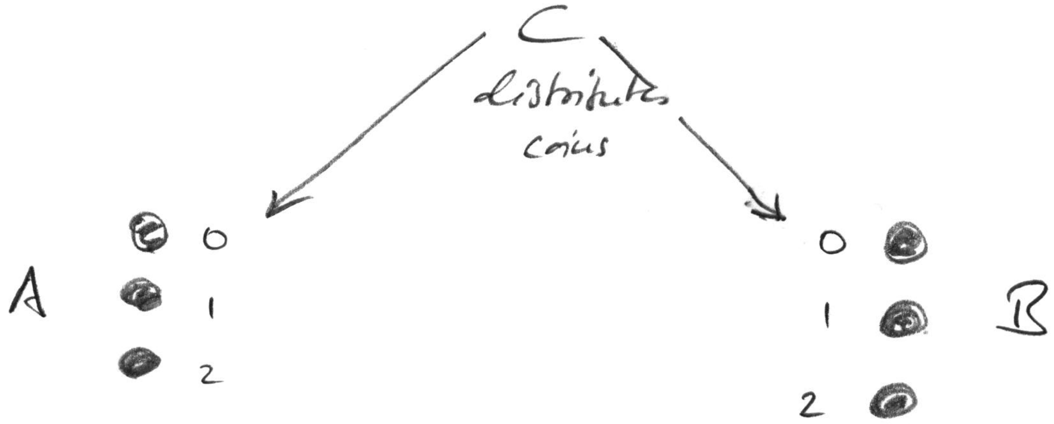
Aims of study of entanglement:

- How non-classical are entangled states?
- What can we do with them? ("resource")
- How can we quantify amount of entanglement?
- How can we transform/manipulate entanglement?
- What about entanglement of mixed states?

2. Bell inequalities

How non-classical are entangled states?

Consider the following game of A+B with coins:



- A+B each get 3 coins in boxes (labelled 0, 1, 2), prepared in some (deterministic or random) way by C.
- A & B can look at one coin each ($i=0,1,2$ & $j=0,1,2$). Result is heads $=+1$ or tails $=-1$. We denote result by $a_i = \pm 1$ and $b_j = \pm 1$.
- A & B observe: If they look at the same coin, they always get the same outcome: $\boxed{a_i = b_i}$

- Can A infer the value of 2 of her coins?

<u>Idea</u>: A looks at i, B at $j = i' \neq i$.

Since $a_{i'} = b_{i'}$, they now know a_i <u>and</u> $a_{i'}$.

Clearly works classically!

- What does this imply?

  - A & B can use this to estimate prob. $p(a_i = a_{i'})\ \forall i, i'$.

  - Clearly, we must have

$$p(a_0 = a_1) + p(a_1 = a_2) + p(a_2 = a_0) \geq 1,$$

since in each instance of the game, at least 2 coins must be equal.

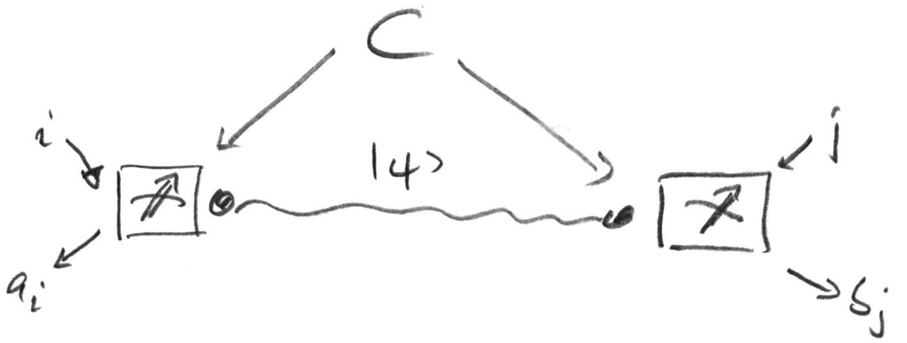
$$\xRightarrow{a_{i'} = b_{i'}} \underline{\underline{p(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) \geq 1}} \quad (*)$$

is satisfied classically!

(*) is called a <u>Bell inequality</u>.

<u>But</u>: In a quantum mechanical version of the game, the Bell inequality (*) can be violated!

Q.M. version of game:



* C distributes an entangled state $|\psi\rangle$.

* A & B perform meas. which depends on $i/j \to a_i/b_j$.

Choose $|\psi\rangle = |\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$.

A & B will measure spin along some axes $\vec{u}_i$ and $\vec{m}_j$, i.e. the operators $\vec{u}_i \cdot \vec{\sigma}^A$ and $\vec{m}_j \cdot \vec{\sigma}^B$, with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

We have $(\vec{\sigma}^A + \vec{\sigma}^B)|\psi^-\rangle = 0$ (i.e. $(\sigma^A_\ell + \sigma^B_\ell)|\psi^-\rangle = 0 \ \forall \ell$)

Then, $\langle\psi^-|(\vec{\sigma}^A \cdot \vec{u})(\underbrace{\vec{\sigma}^B}_{\vec{\sigma}^B|\psi^-\rangle = -\vec{\sigma}^A|\psi^-\rangle} \cdot \vec{m})|\psi^-\rangle =$

$$= -\langle\psi^-|(\vec{\sigma}^A \cdot \vec{u})(\vec{\sigma}^A \cdot \vec{m})|\psi^-\rangle$$

$$= \sum_{k\ell} u_k m_\ell \underbrace{\mathrm{tr}\Big(\underbrace{\rho_A}_{=\frac{1}{2}\mathbb{1}} \sigma_k^A \sigma_\ell^A\Big)}_{=\delta_{k\ell}} = -\sum_k u_k m_k = -\vec{u}\cdot\vec{m}$$

$= -\cos\theta$

angle betw. $\vec{u}$ & $\vec{m}$.

Measurement of A/B along $\vec{n}/\vec{m}$:

$\longrightarrow$ projections $E_{\pm 1}(\vec{u}) = \frac{1}{2}(\mathbb{1} \pm \vec{u}\cdot\vec{\sigma})$

$$p(\pm 1, \pm 1) = \langle \psi^- | E_{\pm 1}(\vec{u})\, E_{\pm 1}(\vec{m}) | \psi^- \rangle$$

$$= \frac{1}{4} \langle \psi^- | \underbrace{\mathbb{1}}_{\hat{=}1} \pm \underbrace{\vec{u}\cdot\vec{\sigma}^A}_{\hat{=}0} \pm \underbrace{\vec{m}\cdot\vec{\sigma}^B}_{\hat{=}0} + \underbrace{(\vec{u}\cdot\vec{\sigma}^A)(\vec{m}\cdot\vec{\sigma}^B)}_{\hat{=}-\cos\Theta} | \psi^- \rangle$$

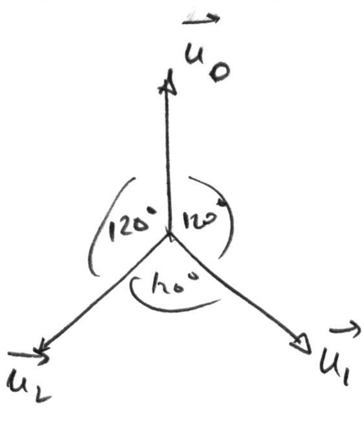
$$= \frac{1}{4}(1 - \cos\Theta)$$

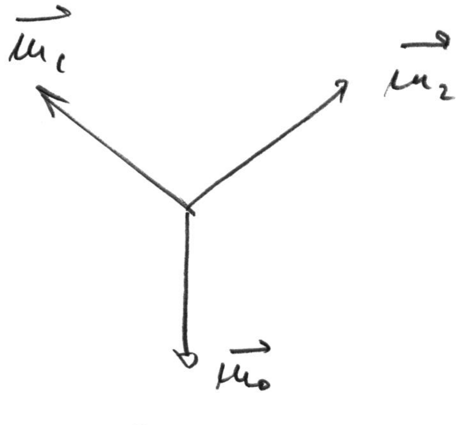
and

$$p(\pm 1, \mp 1) = \frac{1}{4}(1 + \cos\Theta).$$

$\Longrightarrow$ $P_{equal} = \frac{1}{2}(1-\cos\Theta)$, $P_{different} = \frac{1}{2}(1+\cos\Theta)$.

Now let A measure along





in the XZ-plane, and B along $\vec{m}_i = -\vec{u}_i$.

- $i=j$: $P_{\text{equal}} = \frac{1}{2}(1-\cos 180°) = 1$ ✓
  (same basis for A & B)

- $i \neq j$: $P_{\text{equal}} = \frac{1}{2}\Big(1 - \underbrace{\cos(\pm 60°)}_{=\frac{1}{2}}\Big) = \frac{1}{4}$
  (diff. basis for A & B)

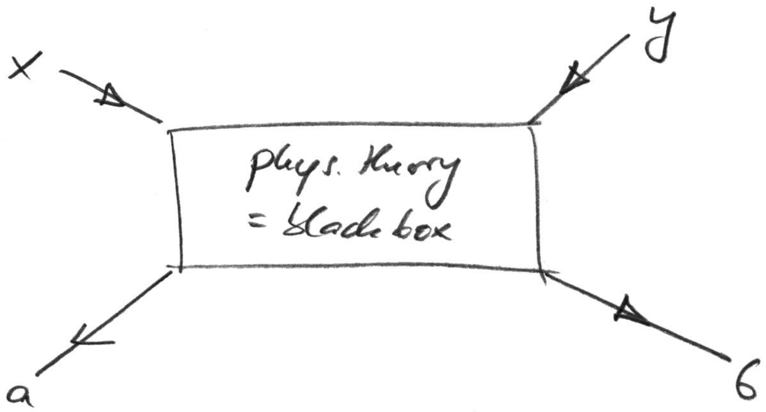
$\Rightarrow$ $P(a_0=b_1) + p(a_1=b_2) + p(a_2=b_0) = \frac{3}{4} < 1$

$\Rightarrow$ Bell inequality <u>violated</u>.

$\Rightarrow$ Q.M. prediction <u>incompatible</u> w/ <u>local realistic</u> description: We cannot assign values to observables we have not measured ("realism").

<u>More Bell inequalities:</u>

Formal setup



x, y: input (meas. setting); a, b: output (outcome of meas.)

Characterized by conditional prob. distribution

$$P(a,b|x,y) \qquad \left(\sum_{a,b} P(a,b|x,y) = 1 \quad \forall x,y \right)$$

Which $P(a,b|x,y)$ are consistent w/ a given phys. theory?

<u>"Classical"</u> physics: <u>Local hidden variable (LHV) model</u>

(local realism)

All outcomes are pre-determined by some "hidden" local variable λ (outcomes exist indep. of meas $\equiv$ realism & no faster-than-light-comm $\equiv$ local):

$$P(a,b|x,y) = \sum_{\lambda} p_\lambda \; P_\lambda^A(a|x) \; P_\lambda^B(b|y)$$

prob. over λ

Consider e.g. $x=0,1$ & $y=0,1$, and outcomes $a_x, b_y = \pm 1$.

Since $a_x = \pm 1$, $b_y = \pm 1$:

$$C = (a_0+a_1)\,b_0 + (a_0-a_1)\,b_1 = \pm 2$$

$$\Rightarrow \quad |\langle C\rangle| \leq \langle |C| \rangle = 2.$$

avg. over P

<u>CHSH inequality</u> (Clauser, Horne, Shimony, Holt)

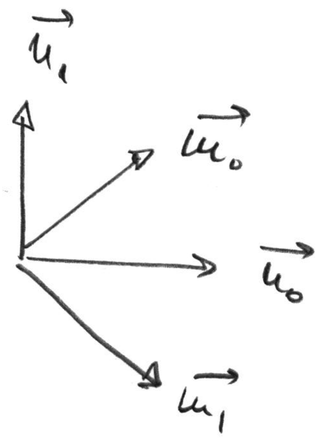
$$\left|\langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle\right| \le 2.$$

<u>Quantum setting:</u>

Ent. state $|\psi^-\rangle$;

$a_x = \vec{\sigma}^A \cdot \vec{u}_x$

$b_y = \vec{\sigma}^B \cdot \vec{u}_y$



$\langle a_x b_y\rangle = -\cos\theta$

$\Rightarrow \quad \langle a_0 b_0\rangle = \langle a_0 b_1\rangle = \langle a_1 b_0\rangle = -\frac{1}{\sqrt{2}}$

$\qquad \langle a_1 b_1\rangle = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \left|\langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle b_0 a_1\rangle - \langle a_1 b_1\rangle\right| = 2\sqrt{2} > 2.$

$\Rightarrow$ incompatible w/ LHV models!

<u>Note:</u> i) Unlike original Bell ineq., no extra info on corr. required!

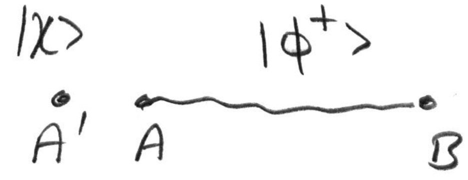
ii) This violation is optimal for Q.M.! <u>But:</u> w/out breaking faster than light, <u>4 is achievable!</u>

($\rightarrow$ HW)

3. Applications of entanglement: Teleportation, dense coding

Teleportation:

Setup:



A & B share ent. state $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$

A has unknown state $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$

(Note: could be part of larger system ($\rightarrow$ linearity!))

A & B cannot send q. states, but can communicate classically "for free"

Can A get $|\chi\rangle$ to B?

Measurement of $|\chi\rangle$ would break state & destroy info ↯

$\Rightarrow$ Teleportation!

(Motivation: Transmitting q. info is subject to noise $\rightarrow$ info could be destroyed. w/ telep., we can first build up $|\phi^+\rangle$ – by retrying or ent. distillation ($\rightarrow$ later) & then teleport: noise-free transmission line!)

Protocol:

1. A performs meas. in Bell basis

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = (Z\otimes I)|\phi^+\rangle = (I\otimes Z)|\phi^+\rangle$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = (X\otimes I)|\phi^+\rangle = (I\otimes X)|\phi^+\rangle$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = (ZX\otimes I)|\phi^+\rangle = (I\otimes XZ)|\phi^+\rangle$$

We also write $|\phi_{\alpha\beta}\rangle = (Z^\alpha X^\beta \otimes I)|\phi^+\rangle = (I\otimes X^\beta Z^\alpha)|\phi^+\rangle$

$(\alpha,\beta = 0,1)$.

Outcome probabilities for $|\phi_{\alpha\beta}\rangle$:

$$\rho_A = tr_B\left[|\phi^+\rangle\langle\phi^+|_{AB}\right] = \frac{1}{2}\mathbb{1}.$$

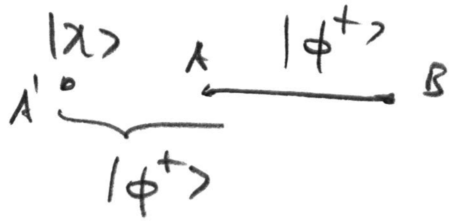
$$\langle\phi_{\alpha\beta}|\, |\chi\rangle\langle\chi|_{A'}\otimes\frac{1}{2}\mathbb{1}_A\, |\phi_{\alpha\beta}\rangle = \frac{1}{2}tr\left[(|\chi\rangle\langle\chi|_{A'}\otimes\mathbb{1})|\phi_{\alpha\beta}\rangle\langle\phi_{\alpha\beta}|\right]$$

$$= \frac{1}{2}tr_{A'}\left[|\chi\rangle\langle\chi|_{A'}\cdot\underbrace{tr_A\left[|\phi_{\alpha\beta}\rangle\langle\phi_{\alpha\beta}|\right]}_{=\frac{1}{2}\mathbb{1}}\right] = \frac{1}{4}.$$

$\Rightarrow$ equal prob. for all 4 outcomes, indep. of $|\chi\rangle$.
(Good: No info about $|\chi\rangle$ acquired $\Rightarrow$ no disturbance.)

What is state of B after meas.?

i) Outcome $|\phi^+\rangle = |\phi_{00}\rangle$:



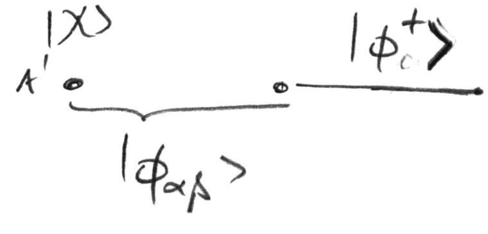
$$\langle\phi^+|_{A'A}\left(|\chi\rangle_{A'}\otimes|\phi^+\rangle_{AB}\right) =$$

$$= \frac{1}{2}\left(\langle 00|_{A'A}+\langle 11|_{A'A}\right)\left(\left(\alpha|0\rangle_{A'}+\beta|1\rangle_{A'}\right)\left(|00\rangle_{AB}+|11\rangle_{AB}\right)\right)$$

$$= \underline{\underline{\frac{1}{2}\left(\alpha|0\rangle_B+\beta|1\rangle_B\right)}}.$$

<u>State $|\chi\rangle$ appears at B!</u>

ii) General outcome:



$$\langle\phi_{\alpha\beta}|_{A'A}\,|\phi^+\rangle_{AB} = \langle\phi^+|_{A'A}\left(\mathbb{1}_{A'}\otimes Z_A^\alpha X_A^\beta\right)|\phi^+\rangle_{AB}$$

$$= \langle\phi^+|_{A'A}\cdot\left(Z_A^\alpha X_A^\beta\otimes\mathbb{1}_B\right)|\phi^+\rangle_{AB}$$

$$= \langle\phi^+|_{A'A}\cdot\left(\mathbb{1}_A\otimes X_B^\beta Z_B^\alpha\right)|\phi^+\rangle_{AB}$$

$$\Rightarrow \left(\langle\phi_{\alpha\beta}|_{A'A}\right)\left(|\chi\rangle_{A'}\otimes|\phi^+\rangle_{AB}\right)$$

$$= X_B^{\beta} Z_B^{\alpha} \cdot \underbrace{\left(\langle\phi_{\alpha\beta}|_{A'A}\,|\chi\rangle_{A'}\otimes|\phi^+\rangle_{AB}\right)}_{=|\chi\rangle_B}.$$

$$= \underline{\underline{X^{\beta}Z^{\alpha}|\chi\rangle}}$$

$\Rightarrow$ outcome is $X^{\beta}Z^{\alpha}|\chi\rangle$ w/ prob. $\frac{1}{4}$ each.

$\Rightarrow$ avg. state of B is $\frac{1}{4}\sum X^{\beta}Z^{\alpha}|\chi\rangle\langle\chi|Z^{\alpha}X^{\beta} = \frac{1}{2}\mathbb{1}$.

$\Rightarrow$ no information at Bob's side!

(2) A communicates meas. outcome (α,β) to Bob, & B applies $(X^{\beta}Z^{\alpha})^{+}$

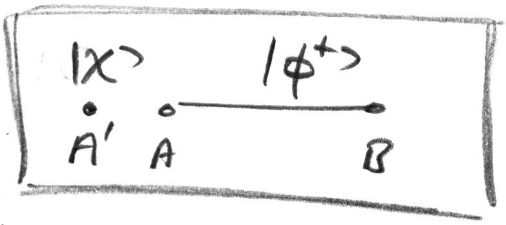
$\Rightarrow$ Bob recovers $|\chi\rangle$.

<u>Notes:</u> * No faster-than-light communication!

* Communicating 1 qubit requires 1 "e-bit" (max. ent. state of 1+1 qubit) + 2 bits of class. communication

Teleportation protocol:

① Measure A, A' in $|\phi_{\alpha\beta}\rangle$ basis.



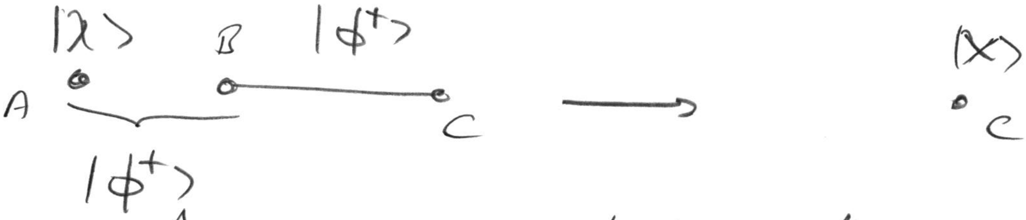
② Communicate (α,β) from A to B.

③ Perform $(X^\beta Z^\alpha)^\dagger$ on B.

Generalization to qu-d-its straightforward! (→ HW)

Relation betw. teleportation & Choi-Jamiolkowski:

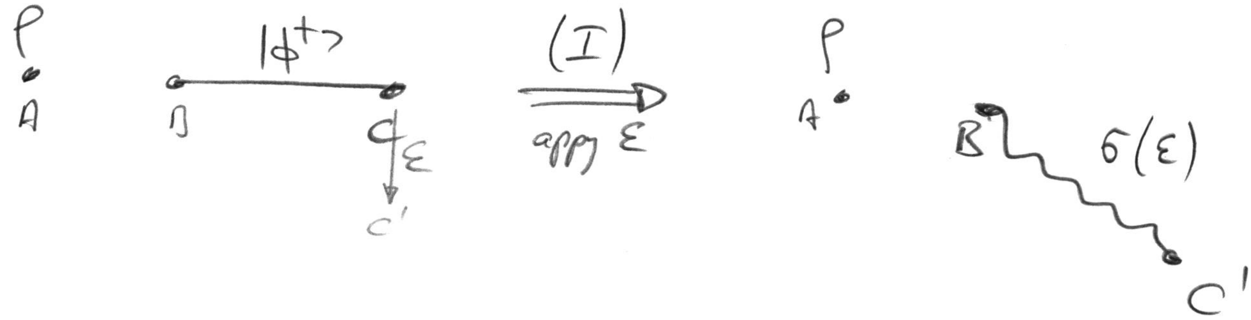
① Consider "postselected teleportation"

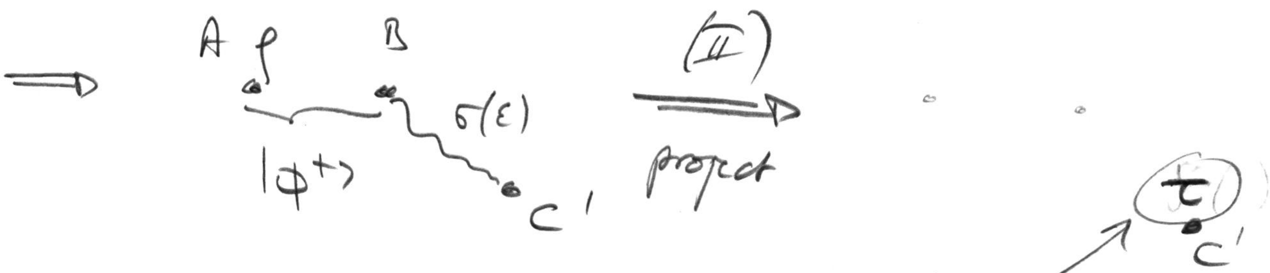


projection onto $|\phi^+\rangle$ ≡ "postselected" meas.

② Protocol for applying $\rho \mapsto \mathcal{E}(\rho)$:

(3) Interchg. order of proj. & appl. of $\mathcal{E}$:





$\boxed{\tau = \mathcal{E}(\rho), \text{ since order commutes!}}$

This is Choi-Jamiołkowski:

(I) is the $\mathcal{E} \longmapsto \sigma$ map, and

(II) is the $\sigma \longmapsto \mathcal{E}$ map.

(check formulas $\rightarrow$ HW)

Dense coding:

Have seen:

- ent. + class. channel $\Rightarrow$ q. channel

  1ebit + 2 class. bits $\longrightarrow$ 1 qubit

- converse possible? q channel $\rightarrow$ class. channel?

trivially by encoding $0 \to |0\rangle$, $1 \to |1\rangle$.

q. channel $\longrightarrow$ class. channel

1 qubit $\longrightarrow$ 1 class. bit

<u>Can we do better w/ entanglement?</u>

$$A \underset{\circ\!\sim\!\sim\!\sim\!\sim\!\sim\!\sim\!\circ}{\;\;|\phi^+\rangle\;\;} B$$

Encode <u>2 bits</u> in $|\phi_{\alpha\beta}\rangle$ (ONB)

① A can encode two bits α, β locally:

$$|\phi_{\alpha\beta}\rangle_{AB} = (Z^\alpha X^\beta \otimes I)|\phi^+\rangle_{AB}$$

② A sends her part of the state to B.

③ B measures in Bell basis $\longrightarrow$ recover α, β.

ent. + q. channel $\longrightarrow$ class. channel

1 ebit + 1 qubit $\longrightarrow$ 2 class. bits

("Dense coding" or "superdense coding")

<u>Note</u>: Together w/ teleportation, this shows optimality of the classical/quantum communication for both protocols.