Notes:

- We don't even need to measure B (outcome is never used again!)
- $H^{\otimes n}$ can be understood as Fourier trafo over $\mathbb{Z}_2^{\times n}$

  $\Rightarrow$ period finding via Fourier trafo (cf. later!)

IV. 3. Grover's algorithm

For many hard computational problems, it is possible to check solution efficiently, but we don't know how to find it. — So-called "NP problems".

Examples: Graph coloring, factoring, 3-SAT, Hamiltonian path, tiling problems, ...

Reformulation:

We can compute $f(x) \in \{0,1\}$; $x \in \{0,1,\dots,N-1\}$

— $f(x)$ is a "verifier" for a solution x;

where $f(x) = 1$ means "solution correct" —

and we want to find some x_0 s.t. $f(x_0) = 1$.

(Can be interpreted as "database search": want to find "marked element" x_0 in an unstructured database.)

Assume for now that $x_0 : f(x_0) = 1$ is unique.
(Generalization: later / homework)

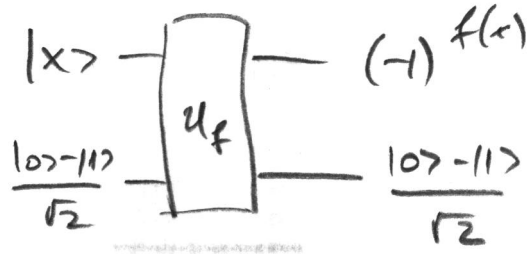
Classically: Need $O(N)$ queries to f for an unstructured search (i.e., w/out using properties of f).

Quantum computer: Will show that $O(\sqrt{N})$ queries enough.
(Note: Only quadratic speedup, but for a very large class of relevant problems)

Ingredient 1:

Oracle $O_f : |x\rangle \longmapsto (-1)^{f(x)} |x\rangle = (-1)^{\delta_{x,x_0}} |x\rangle$

Build via



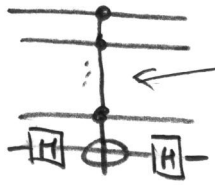
i.e., O_f flips amplitude of "marked" element.

Note that $O_f = I - 2 \cdot |x_0\rangle\langle x_0|$

Ingredient 2:

Unitary $O_0 : |x\rangle \longmapsto (-1)^{\delta_{x,0}} |x\rangle$

Corresponds to $C^{n-1}Z =$



n-qubit Toffoli

→ can be realized efficiently

Again, $O_0 = I - 2|0\rangle\langle 0|$

Define $O_\omega := H^{\otimes n} O_0 H^{\otimes n} = I - 2|\omega\rangle\langle\omega|$; $|\omega\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$.

(Remark: We assumed here $N = 2^n$, but not necessary. Also, every search problem can be inv. embedded s.th. $N = 2^n$.)

Algorithm:

Start from $|\psi_0\rangle = |\omega\rangle = H^{\otimes n}|0\rangle$.

Apply Grover iteration

$$G = -H^{\otimes n} O_0 H^{\otimes n} O_f = -O_\omega O_f :$$

$$|\psi_k\rangle \longmapsto |\psi_{k+1}\rangle = G|\psi_k\rangle = -O_\omega O_f |\psi_k\rangle$$

Observation: Only 2 "special" vectors in O_f, O_ω:

$|x_0\rangle$ and $|\omega\rangle$ $\Rightarrow$ can analyze everything in two-dim. space spanned by $|x_0\rangle$ and $|\omega\rangle$!

Define $|\alpha\rangle := |x_0\rangle$

$$|\beta\rangle := \frac{1}{\sqrt{N-1}} \sum_{x \neq x_0} |x\rangle \propto |\omega\rangle - \frac{1}{\sqrt{N}} |x_0\rangle \Bigg\} |\alpha\rangle \perp |\beta\rangle$$

We can always rewrite

$$a|\alpha\rangle + b|\beta\rangle = x|\omega\rangle + y|\omega^\perp\rangle, \quad \text{with } |\omega^\perp\rangle \perp |\omega\rangle$$

What is effect of O_f and $(-O_\omega)$?

$$O_f \left(a|\alpha\rangle + b|\beta\rangle \right) = -a|\alpha\rangle + b|\beta\rangle$$

$$O_f = I - 2|\alpha\rangle\langle\alpha|$$

$\Rightarrow$ Reflection about $|\beta\rangle$!

$$(-O_\omega)\left(x|\omega\rangle + y|\omega^\perp\rangle \right) = x|\omega\rangle - y|\omega^\perp\rangle$$

$\Rightarrow$ Reflection about $|\omega\rangle$!

I.e.: Grover iteration = 1) reflect about $|\beta\rangle$

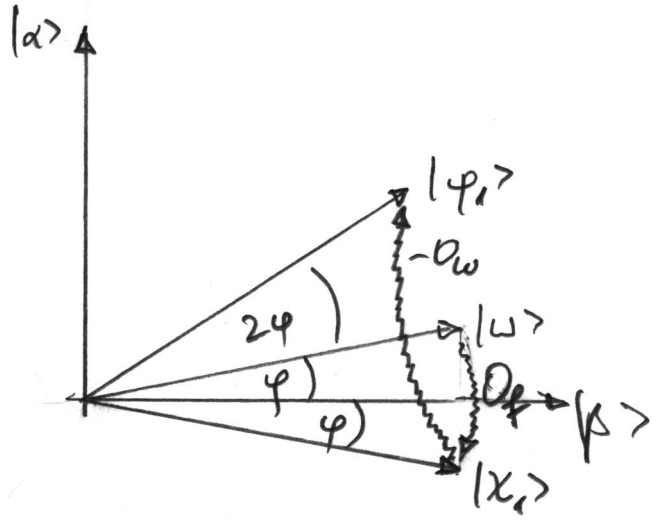
2) reflect about $|\omega\rangle$

So what happens in one iteration, if we start with $|\psi_0\rangle = |\omega\rangle$?

$$|\omega\rangle = \sin\varphi\,|\alpha\rangle + \cos\varphi\,|\beta\rangle$$

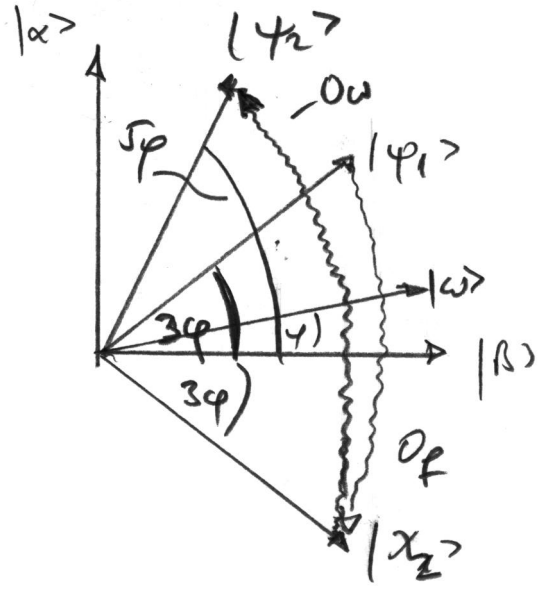
$$|\chi_1\rangle := O_f\,|\omega\rangle$$

$$|\psi_1\rangle = -O_\omega\,|\chi_1\rangle = -O_\omega O_f\,|\omega\rangle$$



$$|\psi_1\rangle = \sin(3\varphi)\,|\alpha\rangle + \cos(3\varphi)\,|\beta\rangle.$$

Next iteration: $|\psi_2\rangle = -O_\omega \underbrace{O_f\,|\psi_1\rangle}_{=:|\chi_2\rangle}$



$$\Rightarrow |\psi_2\rangle = \sin(3\varphi)\,|\alpha\rangle + \cos(3\varphi)\,|\beta\rangle$$

$$\longrightarrow |\psi_k\rangle = \sin\left((2k+1)\varphi\right)|\alpha\rangle + \cos\left((2k+1)\varphi\right)|\beta\rangle .$$

Want that $\varphi_k = (2k+1)\varphi \approx \frac{\pi}{2}$. Then, meas. will w/ high prob. yield $|\alpha\rangle = |k_0\rangle$!

We have: $|\omega\rangle = \frac{1}{\sqrt{N}}|\alpha\rangle + \sqrt{\frac{N-1}{N}}\,|\beta\rangle$

$$= \sin\varphi\,|\alpha\rangle + \cos\varphi\,|\beta\rangle$$

$$\Rightarrow \frac{\sin\varphi}{\cos\varphi} = \frac{\sqrt{\frac{1}{N}}}{\sqrt{\frac{N-1}{N}}} = \frac{1}{\sqrt{N-1}}$$

$$\Rightarrow \varphi \approx \frac{1}{\sqrt{N}} \text{ for large } N.$$

$\Rightarrow$ need $k \approx \frac{\pi}{4}\sqrt{N}$

$\Rightarrow$ $O(\sqrt{N})$ calls to f sufficient!

Quadratic speed-up w.r.t. classical algorithms for general search problems!

Note: o K solutions: Same method works with $O\left(\sqrt{\frac{N}{K}}\right)$ steps ($\rightarrow$ HW)

- Can be adapted to case where K is unknown.

IV.4. The quantum Fourier transform, period finding, and Shor's factoring algorithm

Recall: Simon's algorithm $\rightarrow$ use $H^{\otimes n} \hat{=}$ Fourier trafo over $\mathbb{Z}_2^{\otimes n}$ to find period in $\mathbb{Z}_2^{\otimes n}$.

$\rightarrow$ Can we construct a general Q. Fourier Trafo?

$\rightarrow$ Can it be implemented efficiently?

$\rightarrow$ Applications?

a) The Quantum Fourier Transform (QFT)

Fourier trafo (FT) on $\mathbb{C}^N$:

$$x = (x_0, \dots, x_{N-1}) \in \mathbb{C}^N$$
$$y = (y_0, \dots, y_{N-1}) \in \mathbb{C}^N$$

FT: $x \longmapsto y$ s.th. $y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \cdot e^{2\pi i jk/N}$

Define QFT:

$$\boxed{|j\rangle \longmapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle}$$

(Note: QFT: $\sum x_j |j\rangle \longmapsto \sum x_j e^{2\pi i jk/N} |k\rangle = \sum y_k |k\rangle$

$\Rightarrow$ QFT applies FT to amplitudes)

Computational cost of FT: $O(N^2)$ operations.

With $N = 2^n$ $\rightarrow$ exp. cost in # of bits n.

Fast FT (FFT): $O(N \log N)$, but still $N = \exp(n)$.

Will show: QFT can be implemented in $O(n^2)$ steps

$\longrightarrow$ exponential speed-up!

Rewrite QFT in binary:

* $N = 2^n$
* Write j in binary: $j = j_1 j_2 \dots j_n = j_1 2^{n-1} + j_2 2^{n-2} + \dots + j_n 2^0$
* Decimal part: $0.j_\ell j_{\ell+1} \dots j_n = \frac{1}{2} j_\ell + \frac{1}{4} j_{\ell+1} + \dots$

Then:

$$|j\rangle \longmapsto \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i j k / 2^n} |k\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0,1} \cdots \sum_{k_n=0,1} e^{2\pi i j \left(\sum_{\ell=1}^{n} k_\ell 2^{-\ell}\right)} |k_1, \dots, k_n\rangle$$

$$= \bigotimes_{\ell=1}^{n} \left[\frac{1}{\sqrt{2}} \sum_{k_\ell=0,1} e^{2\pi i j k_\ell 2^{-\ell}} |k_\ell\rangle \right]$$

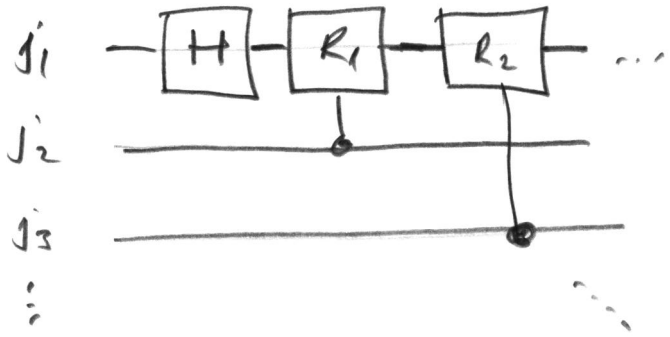
$$= \bigotimes_{\ell=1}^{n} \frac{1}{\sqrt{2}} \left[|0\rangle + e^{2\pi i j 2^{-\ell}} |1\rangle \right]$$

$$\rightarrow j 2^{-\ell} = \underbrace{j_1 j_2 \cdots j_{n-\ell}}_{e^{2\pi i \cdot \text{integer}} = 1} \cdot j_{n-\ell+1} \cdots j_n$$

$$= \frac{|0\rangle + e^{2\pi i 0.j_n} |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{2\pi i 0.j_{n-1} j_n} |1\rangle}{\sqrt{2}} \otimes \dots \otimes \frac{|0\rangle + e^{2\pi i 0.j_1 \dots j_n} |1\rangle}{\sqrt{2}}$$

How to implement this map?

Start w/ rightmost term: $\frac{|0\rangle + e^{2\pi i\, 0.j_1 j_2 \cdots j_n}|1\rangle}{\sqrt{2}}$



$$R_d = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i \cdot 2^{-(d+1)}} \end{pmatrix}$$

action of circuit (up to norm.):

$H_1: |j_1\rangle \longmapsto |0\rangle + e^{2\pi i\, 0.j_1}|1\rangle$

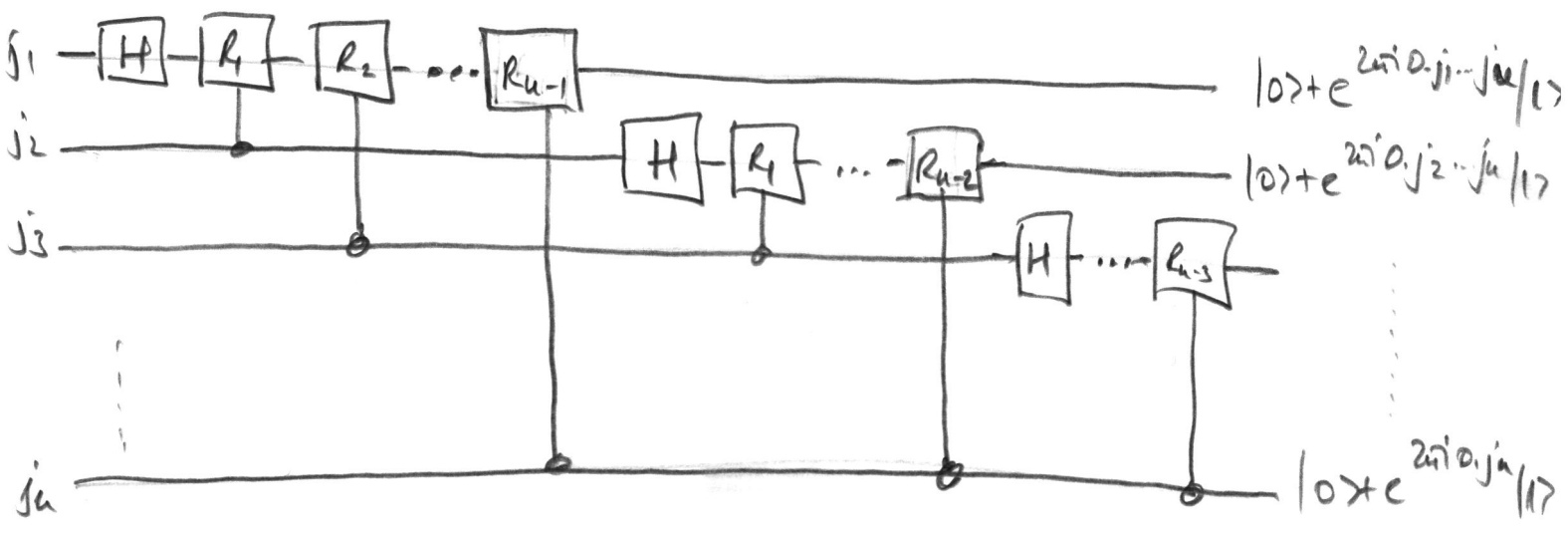
$c\text{-}R_1: (|0\rangle + e^{2\pi i\, 0.j_1}|1\rangle)|j_2\rangle \longmapsto (|0\rangle + e^{2\pi i\, 0.j_1 j_2}|1\rangle)|j_2\rangle$

$c\text{-}R_2: (|0\rangle + e^{2\pi i\, 0.j_1 j_2}|1\rangle)|j_2\rangle|j_3\rangle \longmapsto (|0\rangle + e^{2\pi i\, 0.j_1 j_2 j_3}|1\rangle)|j_2\rangle|j_3\rangle$

$\vdots$ etc.

$\Rightarrow$ obtain n-th Qubit of QFT on 1st qubit.

Continue like that for $(j_2 \cdots j_n)$, $(j_3, \cdots, j_n)$, ...



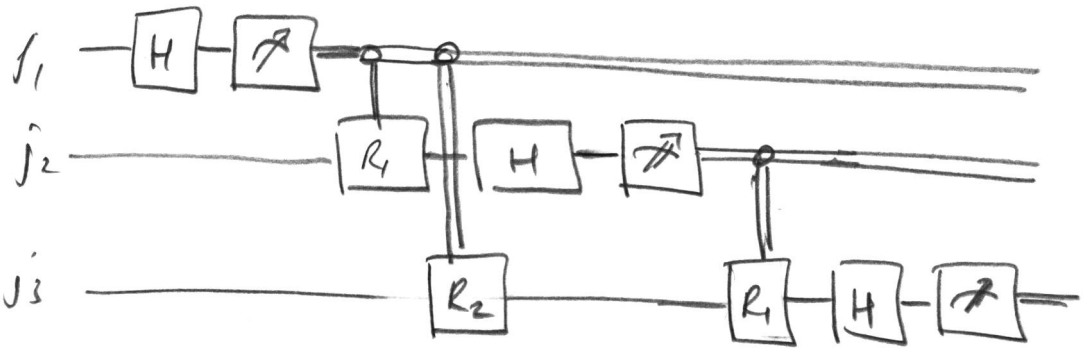
Gate count: $\frac{n(n+1)}{2} = O(n^2)$ gates !

Notes: • Output is in reverse order (but reordering $\sim O(n)$ ops).

•  ⇒ can reverse C-R_d gates.

Upper (control) line acts as control in comp. basis:

If we measure after QFT, we can meas. after H & control R_d-gates classically!



⇒ only one-qubit gates needed!

6) Period finding

Use of QFT: period finding?

Consider $f: \{0,1\}^n \longrightarrow \{0,1\}^n$, s.th. $\exists r > 0$,

$f(x) = f(x+r)$ (and otherwise $f(x) \neq f(y)$)

Can we find r?

Use $U_f : |x\rangle|y\rangle \longmapsto |x\rangle|y \oplus f(x)\rangle$

① $\frac{1}{2^{n/2}} \sum |x\rangle_A |0\rangle_B \xrightarrow{U_f} \frac{1}{2^{n/2}} \sum |x\rangle_A |f(x)\rangle_B$

② Measure $B \to A$ collapses to

$$\frac{1}{\sqrt{k_0}} \sum_{k=0}^{k_0} |x_0 + kr\rangle$$

(Note: As for Simon, we could omit this step!)

③ Apply QFT & measure in comp. basis:

$$\to \frac{1}{2^{n/2}\sqrt{k_0}} \sum_{k}^{k_0} \sum_{\ell=0}^{2^n-1} e^{2\pi i (x_0 + kr)\ell / 2^n} |\ell\rangle$$

$$= \sum_{\ell=0}^{2^n-1} e^{2\pi i x_0 \ell / 2^n} \underbrace{\left[\sum_{k=0}^{k_0-1} \frac{1}{2^{n/2}\sqrt{k_0}} e^{2\pi i k r \ell / 2^n} \right]}_{=: a_\ell} |\ell\rangle$$

$|a_\ell|^2$: probability of outcome ℓ.

If $r \ll 2^n$: many values of k & almost periodic

$\Rightarrow |a_\ell|^2$ peaked around ℓ s.th. $\frac{r\ell}{2^n} \approx$ integer

Explicit analysis of a_ℓ shows: w.h.p., we obtain ℓ s.th.

↳ "with high probability"

$$\frac{\ell}{2^n} \approx \frac{s}{r}$$

If $r \ll 2^n$, this can be used to determine $\frac{s}{r}$ w.h.p.

If s, r are coprime (i.e., $\gcd(s,r) = 1$) – this happens with large enough prob., related to density of primes – we can infer r!

$\Rightarrow$ Quantum Algorithm for period finding!

c) Application: factoring

One use of of period finding: factoring.

Given N (not prime) $\rightarrow$ find non-trivial r: $r \mid N$

↑

"divides"

Algorithm:

① Select random a, $2 \leq a < N$.

If $\gcd(a,N) \geq 1 \Longrightarrow$ done!

↳ eff. computable! (Euclid's algorithm)

So assume $\gcd(a,N) = 1$.

② Let r be the smallest r s.th. $a^r \bmod N = 1$.

(Existence: (!) $\exists x,y: a^x \equiv a^y \bmod N \Rightarrow a^x(1-a^{y-x}) \equiv 0 \bmod N$

$\Rightarrow N \mid a^x(1-a^{y-x}) \overset{\gcd(a,N)=1}{\Longrightarrow} N \mid 1-a^{y-x} \Rightarrow a^{y-x} \equiv 1 \bmod N$)

r is the period of $f_{N,a}(x) = a^x \bmod N$

$f_{N,a}(x)$ can be computed efficiently:

with $x = x_{m-1}2^{m-1} + x_{m-2}2^{m-2} + \dots$,

$$a^x \bmod N = \underbrace{\left(a^{(2^{m-1})}\right)^{x_{m-1}} \cdot \left(a^{(2^{m-2})}\right)^{x_{m-2}} \cdot \dots}_{\text{eff. computable by } a \mapsto a^2 \mapsto a^4 \mapsto a^8 \mapsto \dots} \bmod N,$$

$\Rightarrow r$ can be found eff. w/ a quantum computer!

(3) Assume r even:

$$a^r \bmod N = 1 \iff N \mid (a^r - 1) \iff N \mid (a^{r/2} - 1)(a^{r/2} + 1)$$

and $N \nmid (a^{r/2} - 1)$ (otherwise, $a^{r/2} \bmod N = 1$ ↯)

$\Rightarrow$ either $N \mid a^{r/2} + 1$

or N has non-triv. common factors with both $a^{r/2} \pm 1$

$\Rightarrow 1 \neq \gcd(N, a^{r/2} \pm 1) \mid N$

$\Rightarrow$ found non-triv. factor of N!

$\Rightarrow$ Algorithm successful as long as
(i) r even & (ii) $N \nmid (a^{r/2} + 1)$

Can be shown to happen w/ $p \geq 1/2$ for random choice of a (unless $N = p^k$, p prime → can be checked by taking logs)

$\Rightarrow$ <u>efficient quantum algorithm for factoring</u>

<u>"Shor's algorithm"</u>

V. Quantum Error Correction

V.1. Introduction

- Coupling to environment reduces errors
- Classical computers: "macroscopic" → errors unlikely
- Q. computers: – need qubits = "single" quantum systems
  → fragile!
  – need coupling to "env." to realize gates!

So... can we protect quantum information from noise?

Classical error correction:

copy information, e.g. encode 1 bit in 3:

$$0 \longmapsto \hat{0} = 000$$
$$1 \longmapsto \hat{1} = 111$$

Bit flip w/ some (small) probability $p \Rightarrow$ typ. 0 or 1 bits flipped

Correction: majority vote, i.e.

$$000, 001, 010, 100 \longmapsto 000$$
$$111, 110, 101, 011 \longmapsto 111$$

$p_{error} = \text{prob}(\geq 2 \text{ flips}) = p^3 + 3p^2(1-p) \leq 3p^2 < p$ for $p < 1/3$!

⇒ effective error prob. decreased.

Can be improved by
- → using more bits
- → using smarter codes (encoding k bits)
- → nesting ("concatenating") codes

Quantum error correction:

Several potential problems:

- cannot copy qubits (and even if: how do we compare them?)
- different types of errors, e.g. X (bit flip) or Z (phase flip)
- errors can be continuous
- measuring qubits destroys q. info!

a) The 3-qubit bit flip code

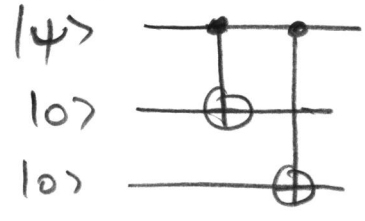
Copy qubits in fixed steps:

$$|0\rangle \longmapsto |\hat{0}\rangle = |000\rangle$$

$$|1\rangle \longmapsto |\hat{1}\rangle = |111\rangle$$

i.e.: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{encoding}} |\hat{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$

Encoding circuit:



$\Rightarrow |\hat{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$

Consider bit flip error $|\hat{\psi}\rangle \mapsto X_i|\hat{\psi}\rangle$ on qubit i;

Can we correct for one bit flip error on unknown qubit?

Problem: Meas. all qubits destroys q. info!

$\Rightarrow$ Need meas. which only returns info about location of error (indep. of encoded state $|\psi\rangle$).

Define "syndrome measurement" with projectors:

"no flip": $P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$

"1st qubit flipped": $P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$

"2nd qubit flipped": $P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$

"3rd qubit flipped": $P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$

Measuring $\{P_\alpha\}$ reveals only 2 bits of information

$\Rightarrow$ one qubit untouched,

By inspection: info. obtained $\equiv$ location of flip, e.g.

$$\alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{qubit 1}]{\text{flip of}} \alpha|100\rangle + \beta|011\rangle$$

$\Rightarrow$ meas. returns P_1, part meas. state

$$\alpha|100\rangle + \beta|011\rangle \xrightarrow[\text{qubit 1}]{\text{recovery: flip}} \alpha|000\rangle + \beta|111\rangle$$

Works for any single or no flipped qubit, & all states $|\psi\rangle$.

(Linearity: also works for parts of large entangled state.)

What about continuous errors, such as

$$|\hat{\psi}\rangle \longmapsto e^{i\vartheta X_i}|\hat{\psi}\rangle = (\cos\vartheta\, \mathbb{1} + i \sin\vartheta\, X_i)|\hat{\psi}\rangle \; ?$$

$$|\hat{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{qubit 1}]{\text{error, eg}} \cos\vartheta \underbrace{(\alpha|000\rangle + \beta|111\rangle)}_{\text{Syndrome } P_0} + i \sin\vartheta \underbrace{(\alpha|100\rangle + \beta|011\rangle)}_{\text{Syndrome } P_1}$$

Syndrome meas. collapses state to:

$p = \cos^2\vartheta$: $P_0 \Rightarrow \alpha|000\rangle + \beta|111\rangle$, no correction ✓

$p = \sin^2\vartheta$: $P_1 \Rightarrow \alpha|100\rangle + \beta|011\rangle$, flip bit 1 ✓

Meas. of error syndrome P_i collapses error onto "digital" error — no error or bit flip $\Rightarrow$ sufficient to study discrete errors!

What about Z errors?

$$\alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{on some qubit}]{\text{Z error}} \alpha|000\rangle - \beta|111\rangle$$

This is still in code space (i.e., space of valid $|\psi\rangle$)

$\Rightarrow$ error not detectable, but $|\psi\rangle$ has changed!

$\Rightarrow$ 3-qubit bit flip code cannot protect against "phase flip" Z.

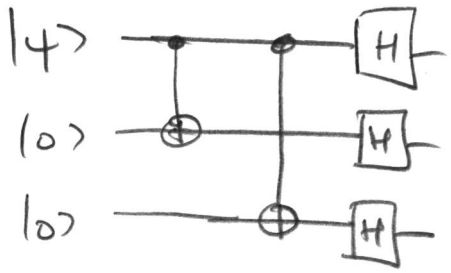
In fact: Phase flip Z_i acts as logical Z on the encoded qubit.

6) 3-qubit phase flip code

We have $Z|+\rangle = |-\rangle$, $Z|-\rangle = |+\rangle$

Z error $\hat{=}$ bit flip error in $|\pm\rangle$-basis.

Encoding $|\hat{0}\rangle = |+++\rangle$, $|\hat{1}\rangle = |---\rangle$ will protect against Z errors!



Syndrome meas. $\tilde{P}_\alpha := H^{\otimes 3} P_\alpha H^{\otimes 3}$, recovery $HX_iH = Z_i$.

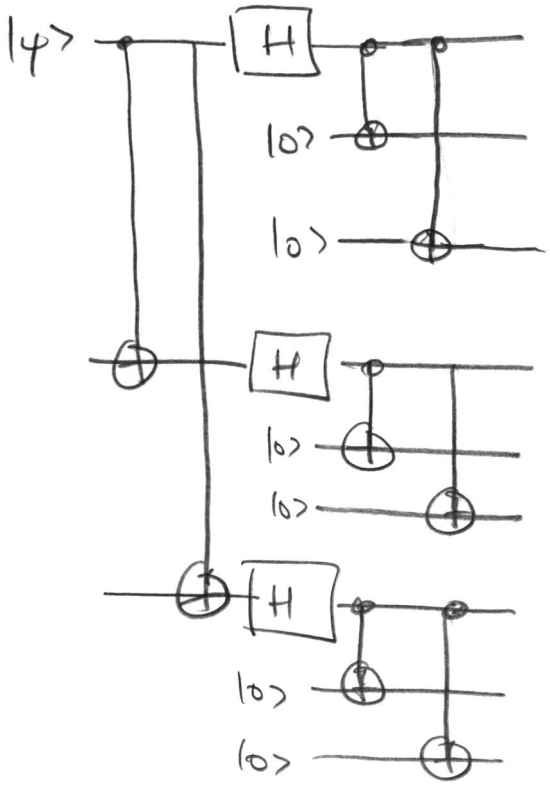
Problem: Now, no protection against bit-flip errors.
(and X_i acts as X on logical qubit)

V.2. The 9-qubit Shor code

Solution: Concatenate (= nest) 3-qubit bit flip with 3-qubit phase flip code!

$$|0\rangle \longmapsto |+\rangle|+\rangle|+\rangle \longmapsto \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \longmapsto |-\rangle|-\rangle|-\rangle \longmapsto \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{2\sqrt{2}}$$



9-qubit Shor code

Can correct any single-qubit Pauli:

(i) X_i error is corrected at "inner" layer.

(ii) Z_i error ≡ logical error on "outer" qubit

⟹ $Z_{block(i)}$ error on "outer" code (phase-flip)

⟹ correctable!

(iii) $Y_i \propto X_i Z_i$: X_i & Z_i corrected independently.