Proof: $|\phi\rangle = \sum \lambda_i |\phi_i^A\rangle \otimes |\phi_i^B\rangle$ (ONBs)

$|\psi\rangle = \sum \lambda_i |\psi_i^A\rangle \otimes |\psi_i^A\rangle$ (ONBs)

$|\phi_i^A\rangle, |\psi_i^A\rangle$ ONB $\Rightarrow \exists U : |\phi_i^A\rangle = U|\psi_i^A\rangle \ \forall i$

& same for B : $\exists V : |\phi_i^B\rangle = V|\psi_i^B\rangle \ \forall i$ $\square$

(Again: Pad w/ 0 if necessary.)

<u>Purification:</u>

Any $|\psi\rangle_{AB}$ s.th. $\mathrm{tr}_B |\psi\rangle\langle\psi|_{AB} = \rho_A$ is called a <u>purification</u> of ρ_A.

$\boxed{\text{need } \underline{\text{not}} \text{ be orthogonal!}}$

(E.g. $\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i| \Rightarrow \sum \sqrt{p_i}\,|\psi_i\rangle|i\rangle$ is purif.)

Given two purifications $|\phi\rangle$ & $|\psi\rangle$ of ρ_A, what is their relation?

Write $|\phi\rangle, |\psi\rangle$ in <u>Schmidt form</u>:

$$|\phi\rangle = \sum \lambda_i |\phi_i^A\rangle |\phi_i^B\rangle \qquad \text{(all ONBs)}$$

$$|\psi\rangle = \sum \mu_i |\psi_i^A\rangle |\psi_i^B\rangle$$

λ_i, μ_i wlog descending.

We have

$$\sum \lambda_i |\phi_i^A\rangle\langle\phi_i^A| = \mathrm{tr}_B |\phi\rangle\langle\phi| = \mathrm{tr}_B |\psi\rangle\langle\psi| = \sum \mu_i |\psi_i^A\rangle\langle\psi_i^A|$$

$\Rightarrow \lambda_i = \mu_i$, $|\phi_i^A\rangle = |\psi_i^A\rangle$ (up to phase)

if λ_i non-degen. (degen. $\rightarrow$ HW)

Now choose $\mathcal{U}$ s.th. $\mathcal{U}|\phi_i^B\rangle = |\psi_i^B\rangle$ $\forall i$ ($\Rightarrow \mathcal{U}$ unitary)

$$\Rightarrow \quad |\psi\rangle = (\mathbb{1} \otimes \mathcal{U})|\phi\rangle.$$

> All purifications are related by a unitary on the purifying system.

(Note: Closely related to unitary equivalence of ensemble decompositions $\rightarrow$ HW!)

36. Mixed states – unitary evolution + projective measurement

Unitary evolution of mixed state

How does a mixed state ρ_A evolve under a unitary $\mathcal{U}_A$?

Consider purification $|\psi\rangle_{AB}$, $\mathrm{tr}_B |\psi\rangle\langle\psi| = \rho_A$.

$$|\psi\rangle \longmapsto (\mathcal{U}_A \otimes \mathbb{1}_B)|\psi\rangle$$

$$\Rightarrow \rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi| \longmapsto \mathrm{tr}_B\left[(U_A \otimes 1\!\!1_B)|\psi\rangle\langle\psi|(U_A^\dagger \otimes 1\!\!1_B)\right]$$

$$= U_A \cdot \mathrm{tr}_B\left[(1\!\!1_A \otimes 1\!\!1_B)|\psi\rangle\langle\psi|(1\!\!1_A \otimes 1\!\!1_B)\right] U_A^\dagger$$

$$= \underline{\underline{U_A \rho_A U_A^\dagger}}$$

(Alt. derivation: $\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i|$ & $|\psi_i\rangle \mapsto U_A|\psi_i\rangle$)

<u>Measurement of mixed states:</u>

Proj. measurement E_n:

Have seen: $p_n = \mathrm{tr}[E_n \rho_A]$.

Post-meas. state:

$$\rho_{A,n} = \frac{1}{p_n} \mathrm{tr}_B\left[(E_n \otimes 1\!\!1)|\psi\rangle\langle\psi|(E_n^\dagger \otimes 1\!\!1)\right]$$

$$= E_n \rho_A E_n^\dagger .$$

5. POVM measurements

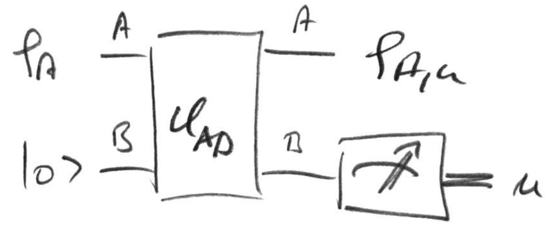
Have seen: add'l system B $\to$ more rich situation

What measurements can we realize by adding extra system?

Idea: i) Add "ancilla" B in state $|0\rangle$

ii) act w/ unitary U_{AB}

iii) measure B in $|0\rangle, \dots, |d_B-1\rangle$



Post-meas. state (un-normalized):

$$\tilde{\rho}_n^A = \langle n|_B U (\rho_A \otimes |0\rangle\langle 0|_B) U^\dagger |n\rangle_B$$

$$= M_n \rho_A M_n^\dagger \text{, with } M_n := \langle n|_B U |0\rangle_B$$

$$\equiv (\mathbb{1}_A \otimes \langle n|_B) U (\mathbb{1}_A \otimes |0\rangle_B)$$

and $\underline{p_n} = \mathrm{tr}\, \tilde{\rho}_n^A = \mathrm{tr}(M_n \rho_A M_n^\dagger) = \underline{\mathrm{tr}(M_n^\dagger M_n \rho_A)}$.

We have

$$\underline{\underline{\sum M_n^\dagger M_n}} = \sum_n \langle 0|_B U^\dagger |n\rangle_B \langle n|_B U |0\rangle_B = \langle 0|_B \mathbb{1}_{AB} |0\rangle_B = \underline{\underline{\mathbb{1}_A}}$$

(Ensures $\sum p_u = \sum \mathrm{tr}(\Pi_u^\dagger \Pi_u \rho_A) = \mathrm{tr}(\rho_A) = 1$)

Definition: A set $\{F_u = \Pi_u^\dagger \Pi_u\}$ with $0 \le F_u \le \mathbb{1}$, $\sum F_u = \mathbb{1}$, is called <u>positive operator-valued measure</u>, and the corresp. measurement w/ outcome probs $p_u = \mathrm{tr}[\Pi_u^\dagger \Pi_u \rho] = \mathrm{tr}[F_u \rho]$ a <u>POVM measurement</u>.

Can any $\{\Pi_u\}$ w/ $\sum \Pi_u^\dagger \Pi_u = \mathbb{1}$ be realized by extensions + unitaries?

$$\begin{pmatrix} \Pi_0 \\ \vdots \\ \Pi_{d-1} \end{pmatrix}$$

$\sum \Pi_u^\dagger \Pi_u = \mathbb{1}$: orth. columns

$\Longrightarrow$

can be extended to unitary

$$U = \begin{array}{c} \\ \langle 0|_B \\ \langle 1|_B \\ \vdots \\ \langle d-1|_B \end{array} \begin{array}{c} |0\rangle_B \quad |1\rangle_B \quad \dots \\ \begin{pmatrix} \Pi_0 & & \\ \Pi_1 & & \\ \vdots & & \\ \Pi_{d-1} & & \end{pmatrix} \end{array}$$

i.e.: $\langle u|_B U |0\rangle_B = \Pi_u$.

$\Rightarrow$ any POVM meas. $\{\Pi_u\}$ can be realized by unitary U + projective measurement!

Is this the <u>most general measurement</u>?

Most gen. linear model: Set $\{F_u\}$ s.th. $p_u = tr[F_u \rho]$.

Wlog., we can choose $F_u = F_u^\dagger$. If not; write

$$F_u = \underbrace{\tfrac{1}{2}(F_u + F_u^\dagger)}_{\text{herm. part}} + \underbrace{\tfrac{1}{2}(F_u - F_u^\dagger)}_{\text{anti-herm. part}}$$

$$tr[(F_u - F_u)^\dagger \rho] \underset{\substack{\uparrow \\ p_u \geq 0:\ \text{trace } \underline{\text{real}}!}}{=} tr[((F_u - F_u^\dagger)\rho)^\dagger] = tr[\rho \cdot (F_u^\dagger - F_u)]$$

$$= - tr[(F_u - F_u^\dagger)\rho]$$

$\Rightarrow tr[(F_u - F_u)^\dagger \rho] = 0 \Rightarrow$ <u>assume F_u hermitian!</u>

<u>Conditions:</u>

- $1 = \sum p_u = tr[(\sum F_u)\rho]$ for all $\rho \Rightarrow \underline{\underline{\sum F_u = \mathbb{1}}}$.
- $0 \leq p_u = tr[F_u \rho]$. $\Rightarrow F_u \geq 0$

  (otherwise choose $|\phi\rangle$ s.th. $F_u|\phi\rangle = \lambda|\phi\rangle$, $\lambda < 0$:

  $$tr[F_u |\phi\rangle\langle\phi|] = \lambda < 0 \ \lightning\ )$$

Moreover, any $F_u \geq 0$ is of the form $F_u = M_u^\dagger M_u$, e.g.

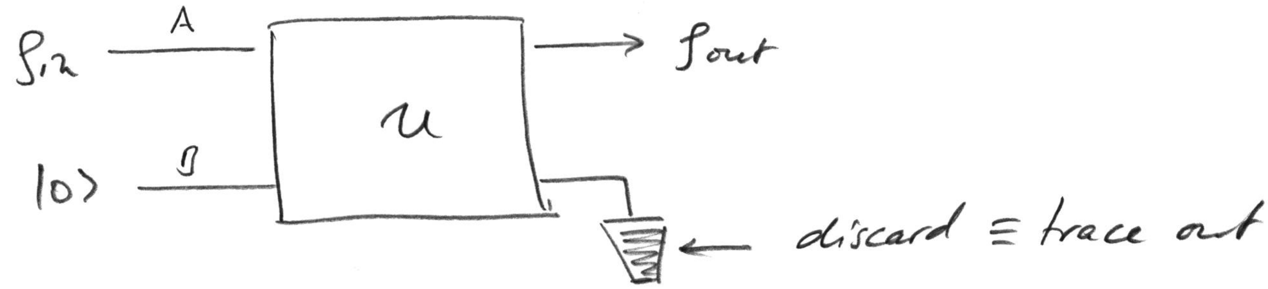
$$F_u = \sum \lambda_i |\phi_i\rangle\langle\phi_i| \Rightarrow M_u = \begin{pmatrix} \sqrt{\lambda_1}\,\langle\phi_1| \\ \sqrt{\lambda_2}\,\langle\phi_2| \\ \vdots \end{pmatrix}$$

$\Rightarrow$ Most general measurement!

6. General evolution – superoperators

Q.: What is the most general physical map on density matrices ("superoperator")?

Idea: Try to add ancilla:



$$\rho \mapsto \mathcal{E}(\rho) = \mathrm{tr}_B\left[U(\rho\otimes|0\rangle\langle 0|)U^\dagger\right]$$

$$= \sum_n \underbrace{\langle n|_B U|0\rangle_B}_{=:\,M_n} \rho \langle 0|_B U^\dagger|n\rangle_B$$

$$= \sum_n M_n \rho M_n^\dagger .$$

(Note: trace in diff. basis $\Rightarrow$ diff. M_n: not unique!)

Properties of M_n? As before:

$$\sum M_n^\dagger M_n = \sum \langle 0|_B U|n\rangle_B \langle n|_B U^\dagger|0\rangle_B = \mathbb{1}_A .$$

Kraus representation:

We call

$$\mathcal{E}(\rho) = \sum M_n \rho M_n^\dagger \; ; \quad \sum M_n^\dagger M_n = \mathbb{1}$$

the <u>Kraus form</u> or <u>Kraus representation</u> of $\mathcal{E}$.

<u>Note</u>: Any such $\mathcal{E}$ can be realized by ancilla + unitary (cf. POVM). In fact, $\mathcal{E}$ can be seen as POVM where we ignore result (meas. by environment).

Is this the most general physical evolution?

<u>Conditions for physical evolution $\mathcal{E}$:</u>

(i) hermiticity-preserving: $\rho = \rho^\dagger \Rightarrow \mathcal{E}(\rho) = \mathcal{E}(\rho)^\dagger$

(ii) positive: $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) \geq 0$.

(iii) trace-preserving: $\mathrm{tr}(\rho) = 1 \Rightarrow \mathrm{tr}(\mathcal{E}(\rho)) = 1$

(iv) linear: $\mathcal{E}(\rho + \lambda\sigma) = \mathcal{E}(\rho) + \lambda\mathcal{E}(\sigma)$

(Note: w/out linearity, ensemble interpretation breaks down $\rightarrow$ HW)

Is this sufficient? <u>NO!</u>

→ $\mathcal{E}$ should still be a physical map when it acts on <u>part of a larger system</u>.

(v) <u>complete positivity</u>:

$$\rho_{AB} \geq 0 \Rightarrow (\mathcal{E}_A \otimes \mathbb{1}_B)(\rho_{AB}) \geq 0$$

(<u>Note</u>: $\mathcal{E}_A \otimes \mathbb{1}_B$ def. on basis: $(\mathcal{E}_A \otimes \mathbb{1}_B)(M \otimes N) = \mathcal{E}_A(M) \otimes N$ + linearity)

We call $\mathcal{E}$ satisfying (i) – (v) a <u>completely positive trace preserving (CPTP) map</u>, or a <u>quantum channel</u>.

Are there maps which are <u>positive</u> ((i)-(iv)) but <u>not CP</u>?

<u>YES</u>: E.g. "transposition channel"

$$\mathcal{E}(\rho) = \rho^T$$

$$(\mathcal{E} \otimes \mathbb{1})(\rho_{AB}) = \rho_{AB}^{T_A} \quad \text{"partial transpose"}$$

E.g.: $|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$$(\mathcal{E}\otimes\mathbb{1})(|\Omega\rangle\langle\Omega|) = \left(|\Omega\rangle\langle\Omega|\right)^{T_B} = \frac{1}{2}\Big[\underbrace{|00\rangle\langle 00|}_{T_B} + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\Big]$$

$$= \frac{1}{2}\begin{pmatrix} 1 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 1 \end{pmatrix} \not\geq 0 \quad !$$

Note: Positive but not CP maps can serve as entanglement witnesses: $(\mathcal{E}\otimes\mathbb{1})(\rho) \geq 0$ for all unentangled states, so $(\mathcal{E}\otimes\mathbb{1})(\sigma) \not\geq 0 \Rightarrow \sigma$ entangled.

$\mathcal{E}$ is in Kraus form $\Rightarrow$ $\mathcal{E}$ CPTP

(by construction or by explicit inspection of

$$(\mathcal{E}\otimes\mathbb{1})(\rho) = \sum \underbrace{(\Pi_n\otimes\mathbb{1})\,\rho\,(\Pi_n\otimes\mathbb{1})^\dagger}_{\geq 0} \quad)$$

Are also all CPTP maps of Kraus form?