Choi-Jamiołkowski isomorphism

Let $\mathcal{C} = \{ \mathcal{E} \mid \mathcal{E} \text{ CPTP} \}$ the space of CPTP maps on the density operators on $\mathbb{C}^d$, and

$$\mathcal{S} := \Big\{ \sigma_{AB} \in \underbrace{\mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)}_{\substack{\text{linear operators} \\ \text{on } \mathbb{C}^d \otimes \mathbb{C}^d}} \;\Big|\; \sigma_{AB} \geq 0,\ \mathrm{tr}_A(\sigma_{AB}) = \frac{1}{d}\mathbb{1} \Big\}.$$

Then, the map

$$\hat{X} : \mathcal{E} \longmapsto \sigma_{AB} = (\mathcal{E}_A \otimes \mathbb{1}_B)(|\Omega\rangle\langle\Omega|), \quad |\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i,i\rangle$$

defines an isomorphism between $\mathcal{C}$ and $\mathcal{S}$, the <u>Choi-Jamiołkowski isomorphism</u>, with σ_{AB} the <u>Choi state</u> of $\mathcal{E}$. The inverse map is given by

$$\hat{Y} : \sigma_{AB} \longmapsto \mathcal{F}; \text{ with } \mathcal{F}(\rho) = d \cdot \mathrm{tr}_B\left[\sigma_{AB} \cdot (\mathbb{1} \otimes \rho^T)\right]$$

<u>Proof</u>: We need to show

(i) $\hat{Y}\hat{X} = \mathbb{1}_{\mathcal{C}}$

(ii) $\hat{X}\hat{Y} = \mathbb{1}_{\mathcal{S}}$

(iii) $\mathrm{Im}\, \hat{X} = \{ \hat{X}(\mathcal{E}) \mid \mathcal{E} \in \mathcal{C} \} \subset \mathcal{S}$

(iv) $\mathrm{Im}\, \hat{Y} \subset \mathcal{C}$.

(i) $\hat{Y}(\underbrace{\hat{X}(\mathcal{E})}_{\equiv \sigma_{AB}})(\rho) = d \cdot \mathrm{tr}_B[\underbrace{\hat{X}(\mathcal{E})}_{\equiv \sigma_{AB}} \cdot (\mathbb{1}_A \otimes \rho^T)]$

$$= \mathrm{tr}_B\left[\left((\mathcal{E} \otimes \mathbb{1}_B)(|\Omega\rangle\langle\Omega|)\right)(\mathbb{1}_A \otimes \rho^T)\right]$$

$$= \sum_{ij} \mathrm{tr}_B\left[\left((\mathcal{E} \otimes \mathbb{1}_B)(|i\rangle\langle j| \otimes |i\rangle\langle j|)\right) \cdot (\mathbb{1}_A \otimes \rho^T)\right]$$

$$= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \cdot \underbrace{\mathrm{tr}_B[|i\rangle\langle j| \rho^T]}_{= \langle j|\rho^T|i\rangle = \rho_{ij}}$$

$$= \mathcal{E}\left(\sum \rho_{ij} |i\rangle\langle j|\right) = \underline{\underline{\mathcal{E}(\rho)}}.$$

✓

(ii) $\hat{X}(\underbrace{\hat{Y}(\sigma_{AB})}_{\equiv \mathcal{F}}) = (\hat{Y}(\sigma_{AB}) \otimes \mathbb{1})(|\Omega\rangle\langle\Omega|)$

$$= \frac{1}{d} \sum_{ij} \underbrace{\left(\hat{Y}(\sigma_{AB})\right)}_{\equiv \mathcal{F}}(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

$$= \frac{1}{\cancel{d}} \sum_{ij} \cancel{d} \cdot \mathrm{tr}_B\left[\sigma_{AB} \cdot (\mathbb{1} \otimes (|i\rangle\langle j|)^T)\right] \otimes |i\rangle\langle j|$$

$$= \sum_{ij} \langle i|_B \, \sigma_{AB} \, |j\rangle_B \otimes |i\rangle_B \langle j|_B = \underline{\underline{\sigma_{AB}}}.$$

✓

(iii) $\sigma_{AB} = \hat{X}(\mathcal{E}) \geq 0$ by construction ($\mathcal{E}$ CPTP).

$$\mathrm{tr}_A(\sigma_{AB}) = \frac{1}{d}\sum_{ij} \underbrace{\mathrm{tr}_A\left[\mathcal{E}(|i\rangle\langle j|)\otimes |i\rangle\langle j|\right]}_{\mathrm{tr}\,\mathcal{E}(|i\rangle\langle j|) = \mathrm{tr}(|i\rangle\langle j|) = \delta_{ij}} = \frac{1}{d}\mathbb{1}_B .$$

$\Rightarrow \sigma_{AB} = \hat{X}(\mathcal{E}) \in \mathcal{S}$ for $\mathcal{E} \in \mathcal{C}$. ✓

(iv) Let $\sigma_{AB} \in \mathcal{S}$. Write $\sigma_{AB} = \sum_k |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|$

unnormalized.

We have $|\tilde{\psi}_k\rangle = \sum_{ij} m_k^{ij} |j\rangle|i\rangle = \frac{1}{\sqrt{d}}\sum_i (\Pi_k \otimes \mathbb{1})|i\rangle|i\rangle$

$$= (\Pi_k \otimes \mathbb{1})|\Omega\rangle$$

$\Rightarrow \sigma_{AB} = \sum_k (\Pi_k \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(\Pi_k \otimes \mathbb{1})^\dagger$, and

$$\left(\hat{Y}(\sigma_{AB})\right)(\rho) = d\,\mathrm{tr}_B\left[\sigma_{AB}\cdot(\mathbb{1}\otimes\rho^T)\right]$$

$$= d\,\mathrm{tr}_B\left[\left(\sum_k (\Pi_k\otimes\mathbb{1})|\Omega\rangle\langle\Omega|(\Pi_k\otimes\mathbb{1})^\dagger\right)(\mathbb{1}\otimes\rho^T)\right]$$

$$= d\sum_k \Pi_k \underbrace{\mathrm{tr}_B\left[|\Omega\rangle\langle\Omega|\cdot(\mathbb{1}\otimes\rho^T)\right]}_{=\frac{1}{d}\sum_{ij}|i\rangle\langle j|_A\,\mathrm{tr}_B[|i\rangle\langle j|\rho^T] = \frac{1}{d}\rho} \Pi_k^\dagger$$

$$= \sum_k \Pi_k \rho \Pi_k^\dagger .$$

and $\frac{1}{d}\mathbb{1} = \mathrm{tr}_A\, \sigma_{AB} = \mathrm{tr}_A \Big[\sum_k (\Pi_k \otimes \mathbb{1}) |\Omega\rangle\langle\Omega| (\Pi_k \otimes \mathbb{1})^\dagger \Big]$

$$= \sum_k \mathrm{tr}_A \Big[(\Pi_k^\dagger \Pi_k \otimes \mathbb{1}) |\Omega\rangle\langle\Omega| \Big]$$

$$= \frac{1}{d} \sum_{ijk} \mathrm{tr}\left(\Pi_k^\dagger \Pi_k\, |i\rangle\langle j| \right) \otimes |i\rangle\langle j|$$

$$\Rightarrow \quad \langle j| \Pi_k^\dagger \Pi_k |i\rangle = \mathrm{tr}\left(\Pi_k^\dagger \Pi_k\, |i\rangle\langle j| \right) = \delta_{ij}$$

$$\Rightarrow \quad \sum_k \Pi_k^\dagger \Pi_k = \mathbb{1}\,.$$

$$\Rightarrow \quad \hat{Y}(\sigma_{AB}) \in \mathcal{C} \text{ for } \sigma_{AB} \in \mathcal{S}. \quad \checkmark$$

$\boxtimes$

<u>Note:</u> The isomorphism still holds if we drop trace preserving and $\mathrm{tr}_A\, \sigma_{AB} = \frac{1}{d}\mathbb{1}$, respectively.

<u>Corollary (from (iv))</u>: All CPTP maps are of Kraus form (and can thus be realized w/ ancilla + unitary + tracing).

7. Axioms ("mixed version")

- States are linear operators with

$$\rho \geq 0$$
$$\mathrm{tr}\rho = 1.$$

- Evolutions are completely positive trace preserving (CPTP) maps

$$\mathcal{E}(\rho) = \sum M_n \rho M_n^\dagger \quad \text{with } \sum M_n^\dagger M_n = \mathbb{1}.$$

- Measurements act as

$$\rho \mapsto \rho_n = \frac{M_n \rho M_n^\dagger}{\mathrm{tr}(M_n \rho M_n^\dagger)},$$

with prob. $p_n = \mathrm{tr}(M_n^\dagger M_n \rho)$ and $\sum M_n^\dagger M_n = \mathbb{1}$.

IV. Entanglement

1. Introduction

Consider a biparthite pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

$|\psi\rangle$ is a product state if it can be written as

$$|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle.$$

If $|\psi\rangle$ cannot be written in this form, we say $|\psi\rangle$ is entangled.

Characterization:

- Schmidt coefficients $\vec{\lambda} = (\lambda_1, \lambda_2, \dots)$

  product state: $\vec{\lambda} = (1, 0, \dots 0)$

  entangled state: $\vec{\lambda} = (\lambda_1, \lambda_2 \neq 0, \dots)$

- Reduced density matrix:

  product state: $\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi| = |\phi_A\rangle\langle\phi_A|$

  $\rho_B = |\phi_B\rangle\langle\phi_B|$.

  $\Rightarrow$ reduced state is pure.

And <u>conversely</u>:

ρ_A pure $\Rightarrow$ Schmidt coeffs $(1, 0, \ldots 0)$

$\Rightarrow |\psi\rangle = |\chi_A\rangle \otimes |\chi_B\rangle$ & $|\chi_A\rangle, |\chi_B\rangle$ determined by ρ_A, ρ_B.

Measured by <u>purity</u>: $\mathrm{tr}\, \rho_A^2 = 1$ for pure ρ_A's or some entropies of Schmidt coeffs.

<u>entangled state</u>:

ρ_A mixed $\Rightarrow \mathrm{tr}\, \rho_A^2 < 1$.

Entangled states "different":

- Cannot describe parts independently
- meas. outcomes correlated
- will see: suitable for non-trivial tasks.

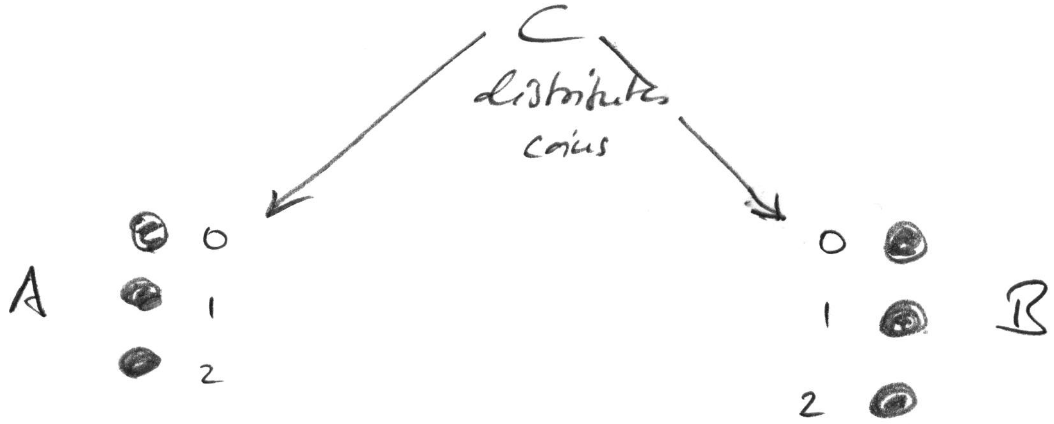
<u>Aims of study of entanglement:</u>

- How <u>non-classical</u> are entangled states?
- What can we do with them? ("resource")
- How can we <u>quantify</u> amount of entanglement?
- How can we transform/manipulate entanglement?
- What about <u>entanglement of mixed states</u>?

2. Bell inequalities

How non-classical are entangled states?

Consider the following game of A+B with coins:



- A+B each get 3 coins in boxes (labelled 0, 1, 2), prepared in some (deterministic or random) way by C.
- A & B can look at one coin each ($i=0,1,2$ & $j=0,1,2$). Result is heads $=+1$ or tails $=-1$. We denote result by $a_i = \pm 1$ and $b_j = \pm 1$.
- A & B observe: If they look at the same coin, they always get the same outcome: $\boxed{a_i = b_i}$

- Can A infer the value of 2 of her coins?

<u>Idea</u>: A looks at i, B at $j = i' \neq i$.

Since $a_{i'} = b_{i'}$, they now know a_i <u>and</u> $a_{i'}$.

Clearly works classically!

- What does this imply?
  - A & B can use this to estimate prob. $p(a_i = a_{i'})\ \forall i, i'$.
  - Clearly, we must have

$$p(a_0 = a_1) + p(a_1 = a_2) + p(a_2 = a_0) \geq 1,$$

since in each instance of the game, at least 2 coins must be equal.

$$\overset{a_{i'} = b_{i'}}{\Longrightarrow} \quad \underline{\underline{p(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) \geq 1}} \quad (*)$$

is satisfied classically!

$(*)$ is called a <u>Bell inequality</u>.

<u>But</u>: In a quantum mechanical version of the game, the Bell inequality $(*)$ can be violated!