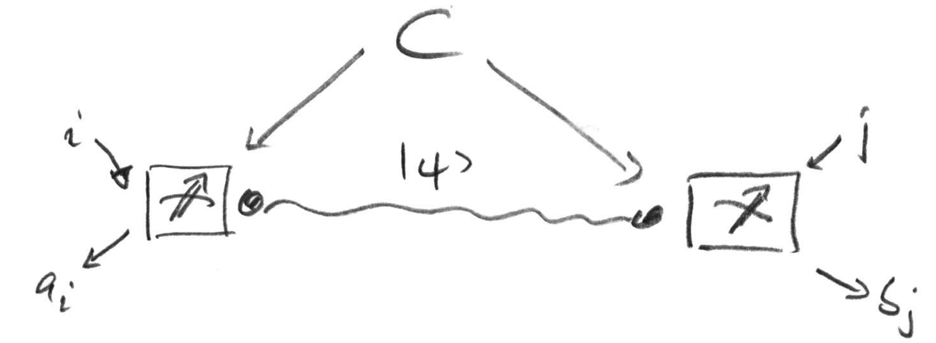
Q.M. version of game:



* C distributes an entangled state $|\psi\rangle$.

* A & B perform meas. which depends on $i/j \rightarrow a_i/b_j$.

Choose $|\psi\rangle = |\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$.

A & B will measure spin along some axes $\vec{u}_i$ and $\vec{m}_j$, i.e. the operators $\vec{u}_i \cdot \vec{\sigma}^A$ and $\vec{m}_j \cdot \vec{\sigma}^B$, with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

We have $(\vec{\sigma}^A + \vec{\sigma}^B)|\psi^-\rangle = 0$ (i.e. $(\sigma^A_\ell + \sigma^B_\ell)|\psi^-\rangle = 0 \ \forall \ell$)

Then, $\langle\psi^-|(\vec{\sigma}^A \cdot \vec{u})(\underbrace{\vec{\sigma}^B}_{\vec{\sigma}^B|\psi^-\rangle = -\vec{\sigma}^A|\psi^-\rangle} \cdot \vec{m})|\psi^-\rangle =$

$$= -\langle\psi^-|(\vec{\sigma}^A \cdot \vec{u})(\vec{\sigma}^A \cdot \vec{m})|\psi^-\rangle$$

$$= \sum_{k\ell} u_k m_\ell \underbrace{\mathrm{tr}\Big(\underbrace{\rho_A}_{=\frac{1}{2}\mathbb{1}} \sigma_k^A \sigma_\ell^A\Big)}_{=\delta_{k\ell}} = -\sum_k u_k m_k = -\vec{u}\cdot\vec{m}$$

$$= -\cos\theta$$

angle betw. $\vec{u}$ & $\vec{m}$.

Measurement of A/B along $\vec{n}/\vec{m}$:

$\longrightarrow$ projections $E_{\pm 1}(\vec{n}) = \frac{1}{2}(\mathbb{1} \pm \vec{n}\cdot\vec{\sigma})$

$$p(\pm 1, \pm 1) = \langle \psi^- | E_{\pm 1}(\vec{n})\, E_{\pm 1}(\vec{m}) | \psi^- \rangle$$

$$= \frac{1}{4} \langle \psi^- | \underbrace{\mathbb{1}}_{\hat{=}1} \pm \underbrace{\vec{n}\cdot\vec{\sigma}^A}_{\hat{=}0} \pm \underbrace{\vec{m}\cdot\vec{\sigma}^B}_{\hat{=}0} + \underbrace{(\vec{n}\cdot\vec{\sigma}^A)(\vec{m}\cdot\vec{\sigma}^B)|\psi^-\rangle}_{\hat{=}-\cos\Theta}$$

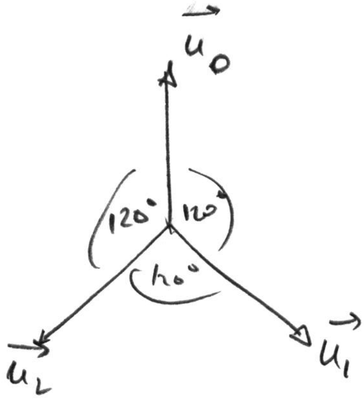
$$= \frac{1}{4}(1-\cos\Theta)$$

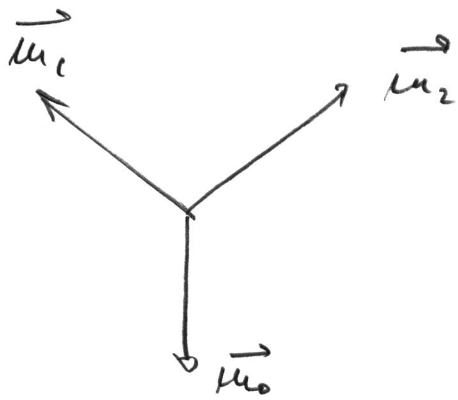
and

$$p(\pm 1, \mp 1) = \frac{1}{4}(1+\cos\Theta).$$

$\Longrightarrow$ $P_{equal} = \frac{1}{2}(1-\cos\Theta)$, $P_{different} = \frac{1}{2}(1+\cos\Theta)$.

Now let A measure along





in the XZ-plane, and B along $\vec{m}_i = -\vec{u}_i$.

- $i = j$: (same basis for A & B) $\quad P_{equal} = \frac{1}{2}(1 - \cos 180°) = 1$ ✓

- $i \neq j$: (diff. basis for A & B) $\quad P_{equal} = \frac{1}{2}\Big(1 - \underbrace{\cos(\pm 60°)}_{=\frac{1}{2}}\Big) = \frac{1}{4}$

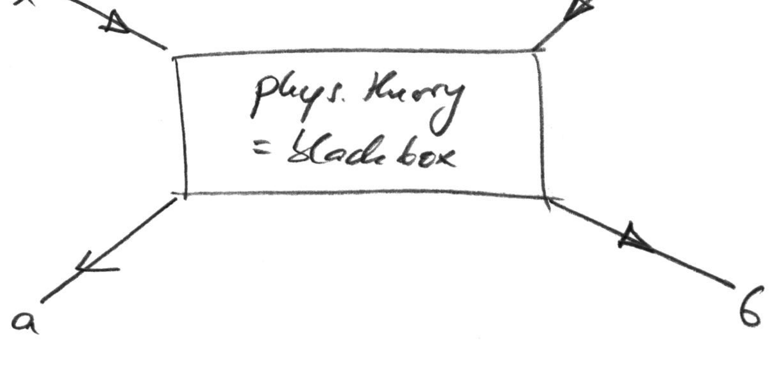
$$\Rightarrow \quad P(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) = \frac{3}{4} < 1$$

$\Rightarrow$ Bell inequality <u>violated</u>.

$\Rightarrow$ Q.M. prediction <u>incompatible</u> w/ <u>local realistic</u> description: We cannot assign values to observables we have not measured ("realism").

<u>More Bell inequalities:</u>

Formal setup



x, y: input (meas. setting); a, b: output (outcome of meas.)

Characterized by conditional prob. distribution

$$P(a,b|x,y) \qquad \left(\sum_{a,b} P(a,b|x,y) = 1 \quad \forall x,y \right)$$

Which $P(a,b|x,y)$ are consistent w/ a given phys. theory?

"Classical" physics: Local hidden variable (LHV) model
(local realism)

All outcomes are pre-determined by some "hidden" local variable λ (outcomes exist indep. of meas $\equiv$ realism & no faster-than-light-comm. $\equiv$ local):

$$P(a,b|x,y) = \sum_\lambda \underbrace{p_\lambda}_{\text{prob. over } \lambda} P_\lambda^A(a|x)\, P_\lambda^B(b|y)$$

Consider e.g. $x=0,1$ & $y=0,1$, and outcomes $a_x, b_y = \pm 1$.

Since $a_x = \pm 1$, $b_y = \pm 1$:

$$C = (a_0 + a_1) b_0 + (a_0 - a_1) b_1 = \pm 2$$

$$\Rightarrow \underbrace{|\langle C \rangle|}_{\text{avg. over } P} \leq \langle |C| \rangle = 2 \, .$$

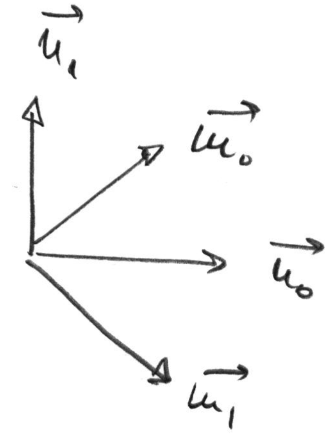
CHSH inequality (Clauser, Horne, Shimony, Holt)

$$\left|\langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle\right| \le 2.$$

Quantum setting:

Ent. state $|\psi^-\rangle$;

$$a_x = \vec{\sigma}^A \cdot \vec{u}_x$$
$$b_y = \vec{\sigma}^B \cdot \vec{u}_y$$



$$\langle a_x b_y\rangle = -\cos\theta$$

$$\Rightarrow \quad \langle a_0 b_0\rangle = \langle a_0 b_1\rangle = \langle a_1 b_0\rangle = -\frac{1}{\sqrt{2}}$$
$$\langle a_1 b_1\rangle = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \left|\langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle b_0 a_1\rangle - \langle a_1 b_1\rangle\right| = 2\sqrt{2} > 2.$$

$\Rightarrow$ incompatible w/ LHV models!

Note: i) Unlike original Bell ineq., no extra info on corr. required!

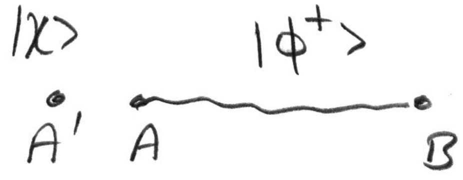
ii) This violation is optimal for Q.M! But: w/out breaking faster than light, 4 is achievable!

($\rightarrow$ HW)

3. Applications of entanglement: Teleportation, dense coding

Teleportation:

Setup:



A & B share ent. state $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$

A has unknown state $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$

(Note: could be part of larger system ($\rightarrow$ linearity!))

A & B cannot send q. states, but can communicate classically "for free"

Can A get $|\chi\rangle$ to B?

Measurement of $|\chi\rangle$ would break state & destroy info ↯

$\Rightarrow$ Teleportation!

(Motivation: Transmitting q. info is subject to noise $\rightarrow$ info could be destroyed. w/ telep., we can first build up $|\phi^+\rangle$ – by retrying or ent. distillation ($\rightarrow$ later) & then teleport: noise-free transmission line!)

Protocol:

① A performs meas. in <u>Bell basis</u>

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = (Z\otimes I)|\phi^+\rangle = (I\otimes Z)|\phi^+\rangle$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = (X\otimes I)|\phi^+\rangle = (I\otimes X)|\phi^+\rangle$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = (ZX\otimes I)|\phi^+\rangle = (I\otimes XZ)|\phi^+\rangle$$

We also write $|\phi_{\alpha\beta}\rangle = (Z^\alpha X^\beta \otimes I)|\phi^+\rangle = (I\otimes X^\beta Z^\alpha)|\phi^+\rangle$

$(\alpha, \beta = 0, 1)$.

Outcome probabilities for $|\phi_{\alpha\beta}\rangle$:

$$\rho_A = \mathrm{tr}_B\left[|\phi^+\rangle\langle\phi^+|_{AB}\right] = \frac{1}{2}\mathbb{1}.$$

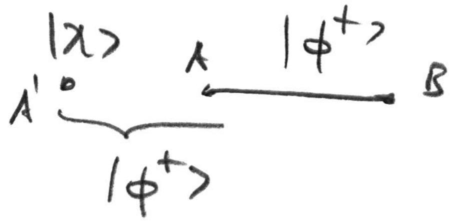
$$\langle\phi_{\alpha\beta}|\, |\chi\rangle\langle\chi|_{A'} \otimes \frac{1}{2}\mathbb{1}_A\, |\phi_{\alpha\beta}\rangle = \frac{1}{2}\mathrm{tr}\left[(|\chi\rangle\langle\chi|_{A'}\otimes\mathbb{1})|\phi_{\alpha\beta}\rangle\langle\phi_{\alpha\beta}|\right]$$

$$= \frac{1}{2}\mathrm{tr}_{A'}\left[|\chi\rangle\langle\chi|_{A'}\cdot \underbrace{\mathrm{tr}_A\left[|\phi_{\alpha\beta}\rangle\langle\phi_{\alpha\beta}|\right]}_{=\frac{1}{2}\mathbb{1}}\right] = \frac{1}{4}.$$

$\Rightarrow$ equal prob. for all 4 outcomes, <u>indep. of $|\chi\rangle$</u>.
(Good: No info about $|\chi\rangle$ acquired $\Rightarrow$ no disturbance.)

What is state of B after meas.?

i) Outcome $|\phi^+\rangle = |\phi_{00}\rangle$:



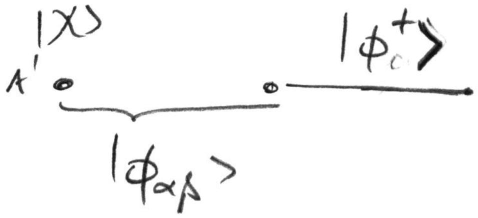
$$\langle \phi^+|_{A'A} \left(|\chi\rangle_{A'} \otimes |\phi^+\rangle_{AB} \right) =$$

$$= \frac{1}{2} \left(\langle 00|_{A'A} + \langle 11|_{A'A} \right) \left(\left(\alpha |0\rangle_{A'} + \beta |1\rangle_{A'} \right) \left(|00\rangle_{AB} + |11\rangle_{AB} \right) \right)$$

$$= \underline{\underline{\frac{1}{2} \left(\alpha |0\rangle_B + \beta |1\rangle_B \right)}}.$$

<u>State $|\chi\rangle$ appears at B!</u>

ii) General outcome:



$$\langle \phi_{\alpha\beta}|_{A'A} \, |\phi^+\rangle_{AB} = \langle \phi^+|_{A'A} \left(\mathbb{1}_{A'} \otimes Z_A^{\alpha} X_A^{\beta} \right) |\phi^+\rangle_{AB}$$

$$= \langle \phi^+|_{A'A} \cdot \left(Z_A^{\alpha} X_A^{\beta} \otimes \mathbb{1}_B \right) |\phi^+\rangle_{AB}$$

$$= \langle \phi^+|_{A'A} \cdot \left(\mathbb{1}_A \otimes X_B^{\beta} Z_B^{\alpha} \right) |\phi^+\rangle_{AB}$$

$$\Rightarrow \left(\langle\phi_{\alpha\beta}|_{A'A}\right)\left(|\chi\rangle_{A'}\otimes|\phi^+\rangle_{AB}\right)$$

$$= X_B^{\beta} Z_B^{\alpha}\cdot \underbrace{\left(\langle\phi_{\alpha\beta}|_{A'A}\,|\chi\rangle_{A'}\otimes|\phi^+\rangle_{AB}\right)}_{=|\chi\rangle_B}.$$

$$= \underline{\underline{X^{\beta}Z^{\alpha}|\chi\rangle}}$$

$\Rightarrow$ outcome is $X^{\beta}Z^{\alpha}|\chi\rangle$ w/ prob. $\frac{1}{4}$ each.

$\Rightarrow$ avg. state of B is $\frac{1}{4}\sum X^{\beta}Z^{\alpha}|\chi\rangle\langle\chi|Z^{\alpha}X^{\beta} = \frac{1}{2}\mathbb{1}$.

$\Rightarrow$ no information at Bob's side!

(2) A communicates meas. outcome (α,β) to Bob, & B applies $(X^{\beta}Z^{\alpha})^{\dagger}$

$\Rightarrow$ Bob recovers $|\chi\rangle$.

<u>Notes:</u> * No faster-than-light communication!

* Communicating 1 qubit requires 1 "e-bit" (max. ent. state of 1+1 qubit) + 2 bits of class. communication