

Requirements for asymptotic protocols:

(67)

• Convert $|\phi^{\otimes n}\rangle \leftrightarrow |\chi^{\otimes n}\rangle$ with rate $\frac{n}{m} \rightarrow R > 0$

for $n, m \rightarrow \infty$.

• success probability $p \rightarrow 1$ for $n \rightarrow \infty$.

• conversion can be imperfect, as long as error $\rightarrow 0$

as $n \rightarrow \infty$.

Error measure: $\delta = 1 - F$; $F = |\langle \psi | \phi \rangle|^2$: "fidelity"

Good measure: Bounds distance for any observable!

What is form of $|\chi^{\otimes n}\rangle$, $|\chi\rangle = \sum \sqrt{p(x)} |x\rangle_A |x\rangle_B$, $x=1, \dots, d$?

$$|\chi^{\otimes n}\rangle = \sum_{x_1, \dots, x_n} \sqrt{p(x_1) \dots p(x_n)} |x_1, \dots, x_n\rangle_A |x_1, \dots, x_n\rangle_B$$

\Rightarrow prob. of $|x_1, \dots, x_n\rangle$: $p(x_1, \dots, x_n) = p(x_1) \dots p(x_n)$:

i.i.d. (independently & identically distributed):

law of large numbers etc. applies!

(i.e., $\text{prob}(|\frac{1}{n} \sum x_i - E(x_i)| \geq \epsilon) \rightarrow 0 \forall \epsilon$)

What is typical output of iid source (i.e., typ $\langle x_1, \dots, x_N \rangle$)? (68)

Most likely: Output x appears $\approx N \cdot p(x)$ times.

$$\Rightarrow P(x_1, \dots, x_N) \approx P(x_1) \cdots P(x_N) \approx p(1)^{Np(1)} \cdots p(d)^{Np(d)}$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{base 2}}}{-\log} P(x_1, \dots, x_N) \approx N \underbrace{\left(- \sum_x p(x) \log p(x) \right)}_{=: H(p): \text{ Shannon entropy of } p}$$

Asymptotically: prob. = 1 to be ϵ -close to this, more

precisely: $\text{prob} \left(\left| -\frac{1}{N} \log P(x_1, \dots, x_N) - H(p) \right| \geq \epsilon \right) \rightarrow 0$

We call all such (x_1, \dots, x_N) ϵ -typical sequences.

There are asymptotically $\approx 2^{NH(p)}$ typ. sequences.

Fix $\epsilon > 0$. Define

$$|\mathcal{D}_N\rangle := \sum_{x_1, \dots, x_N \text{ } \epsilon\text{-typ.}} \sqrt{P(x_1) \cdots P(x_N)} |x_1, \dots, x_N\rangle |x_1, \dots, x_N\rangle,$$

$$|\hat{\mathcal{D}}_N\rangle = \frac{|\mathcal{D}_N\rangle}{\| |\mathcal{D}_N\rangle \|}.$$

We have

$$\langle \hat{\rho}_N | \chi^{\otimes N} \rangle = \frac{\sum_{\epsilon\text{-typ}} P(x_1, \dots, x_N)}{\sqrt{\sum_{\epsilon\text{-typ}} P(x_1, \dots, x_N)}} \xrightarrow{N \rightarrow \infty} 1$$

and # terms $\approx 2^{N H(\rho)}$ (and in fact $\leq 2^{N(H(\rho) + \epsilon)}$).

Protocol $|\phi^+\rangle^{\otimes n} \rightarrow |\chi\rangle^{\otimes N}$:

- Use $M = N(H(\rho) + \epsilon)$ Bell pairs to prepare $|\hat{\rho}_N\rangle$ (possible: $|\phi^+\rangle^{\otimes n}$ universal for all distributions).
- $\frac{M}{N} \rightarrow H(\rho) + \epsilon \rightarrow H(\rho)$, and $|\hat{\rho}_N\rangle \rightarrow |\chi\rangle^{\otimes N}$
 \Rightarrow can prepare $|\chi\rangle$ asymptotically at a cost $H(\rho)$ per copy!

Protocol $|\chi\rangle^{\otimes N} \rightarrow |\phi^+\rangle^{\otimes n}$:

- Use $|\hat{\rho}_N\rangle$ instead of $|\chi\rangle^{\otimes N}$, since fidelity $\rightarrow 1$.
- Schmidt coeffs. approach flat distribution with $N(H(\rho) - \epsilon)$ terms asymptotically
 \Rightarrow Can extract $\frac{n}{N} (= H(\rho) - \epsilon) \rightarrow H(\rho)$ e-bits per copy of $|\chi\rangle$.

Asymptotically: Can dilate $(|\psi^+\rangle^{\otimes n} \rightarrow |\chi\rangle^{\otimes n})$

(70)

and distill $(|\chi\rangle^{\otimes n} \rightarrow |\psi^+\rangle^{\otimes n})$ at the same rate

$H(p)$, with $p = (p_1, \dots, p_d)$, $\sqrt{p_k}$ the Schmidt coeffs.

(Note: Same rate: necessarily optimal!)

Can be expressed in terms of the

$$\boxed{\text{"von Neumann entropy"} \quad S(p) := -\text{tr}(p \log p)}$$

$$H(p) = S(\text{tr}_A |\psi\rangle\langle\psi|) = S(\text{tr}_B |\psi\rangle\langle\psi|)$$

Protocol allows to go reversibly between any two states

$|\psi\rangle^{\otimes K} \leftrightarrow |\chi\rangle^{\otimes L}$ as long as $K S(\text{tr}_A |\psi\rangle\langle\psi|) = L S(\text{tr}_B |\chi\rangle\langle\chi|)$,
by going via $|\psi^+\rangle$.

Result: The entropy of entanglement

$$E(|\psi\rangle) := S(\text{tr}_A |\psi\rangle\langle\psi|) = S(\text{tr}_B |\psi\rangle\langle\psi|)$$

uniquely quantifies the amount of
entanglement in a pure bipartite state.

III.5 Mixed state entanglement

(71)

a) Introduction

When is a mixed state ρ_{AB} entangled?

- i) If ρ_{AB} cannot be created by LOCC (or $\rho_{AB}^{\otimes N}$)
- ii) If we can extract ebits from ρ_{AB} or $\rho_{AB}^{\otimes N}$
- (iii) If it helps us to perform some task better in an LOCC setting.

Clearly, there are more states w/ (i) than (iii) than (ii)
"≥" "≥"

Use (i) as our definition:

States which can be prepared by LOCC:

$$(*) \quad \rho = \sum p_i \rho_i^A \otimes \rho_i^B \quad \text{"separable state"}$$

$$(p_i^A, p_i^B \geq 0, p_i \geq 0; \sum p_i = 1, \sum p_i^A = \sum p_i^B = 1)$$

We define

ρ is entangled if it is not separable

(i.e., it cannot be written in the form (*).

Given ρ , how can we check if it is sep./entangled? (72)

Problem: Need to check over all decompositions

$$\rho \stackrel{?}{=} \sum p_i \rho_i^A \otimes \rho_i^B$$

Arbitrarily n ensemble decomposition = isometries:
need to optimize over isometries!

(Note: can always choose ρ_i pure by further decomp. ρ_i).

In fact, general problem is NP-hard ("exponentially"
hard in dimension of space)

6) Entanglement witnesses

Structure of set of separable states:

$$\text{let } \rho = \sum p_i \rho_i^A \otimes \rho_i^B ; \quad \sigma = \sum q_j \sigma_j^A \otimes \sigma_j^B$$

$$\Rightarrow \lambda \rho + (1-\lambda)\sigma = \sum r_k \chi_k^A \otimes \chi_k^B, \quad \lambda \in [0;1]$$

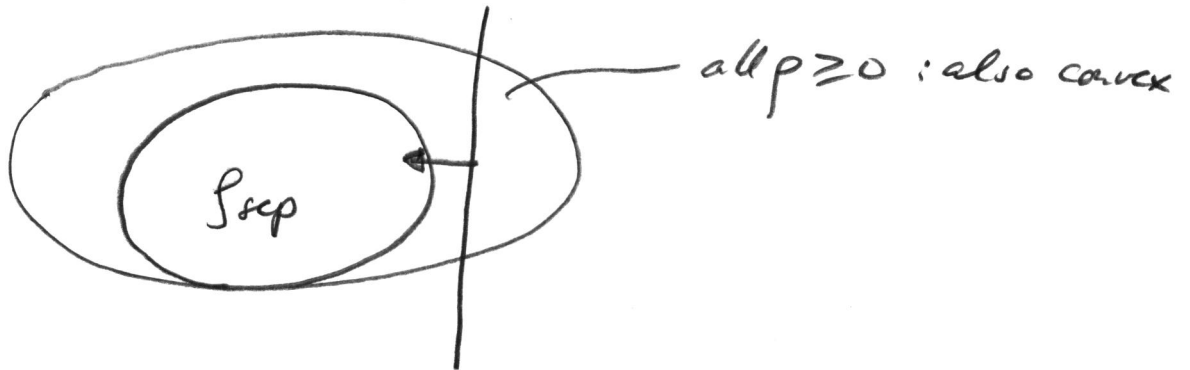
$$\text{with } r_k = (\lambda p_1, \lambda p_2, \dots, (1-\lambda)q_1, (1-\lambda)q_2, \dots)$$

$$\text{and } \chi_k^{A/B} = (p_1^{A/B}, p_2^{A/B}, \dots, q_1^{A/B}, q_2^{A/B}, \dots)$$

$\Rightarrow \lambda \rho + (1-\lambda) \sigma$ separable

(73)

\Rightarrow separable states form convex set!



Can find hyperplanes s.t. all ρ on one side are entangled!

General hyperplane is of form $\text{tr}[X\rho] + c = 0$.

$$\Leftrightarrow \text{tr}[(X + c\mathbb{1})\rho] = 0.$$

$=: W \rightarrow$ can be Hermitian!

Two sides of hyperplane $\Leftrightarrow \text{tr}[W\rho] \geq 0$.

Entanglement witness: $W = W^\dagger$ s.t.

$$\text{tr}[W\rho] \geq 0 \quad \forall \rho \text{ separable, i.e.}$$

$$\text{tr}[W\rho] < 0 \Rightarrow \rho \text{ entangled!}$$

Notes:

- Need ways to show that ρ sep. $\Rightarrow \text{tr}[W\rho] \geq 0$!
- Witness can only detect certain ent. states!
- Convex set char. by all tangent planes:
 $\Rightarrow \exists$ witness for any ent. state.
- Witness linear operator \Rightarrow experimentally measurable!

Example:

$$W = \mathbb{F} \text{ ("flip")}; \quad \mathbb{F} := \sum_{i,j=1}^d |i,j\rangle\langle j,i|$$

"magic formula": $\text{tr}[\mathbb{F}(A \otimes B)] = \sum_{i,j} \text{tr}[|i,j\rangle\langle j,i| A \otimes B]$

$$= \sum \langle j,i| A \otimes B |i,j\rangle = \sum A_{ji} B_{ij} = \text{tr}(AB)$$

$$\text{let } \rho_{\text{sep}} = \sum p_i \rho_i^A \otimes \rho_i^B;$$

$$\text{tr}[W\rho_{\text{sep}}] = \sum p_i \text{tr}(\mathbb{F}(\rho_i^A \otimes \rho_i^B))$$

$$= \sum p_i \text{tr}[\rho_i^A \rho_i^B] \geq 0$$

$$\geq 0: P_i \geq 0, P = \sum \lambda_i |\varphi_i\rangle\langle\varphi_i|;$$

$$\Rightarrow \text{tr}[PQ] = \sum_i \lambda_i \langle\varphi_i| Q |\varphi_i\rangle \geq 0.$$

$\Rightarrow W$ is ent. witness.

(75)

Which states does W detect? $\rightarrow W$ detect anti-sym. states!

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \Rightarrow F|\psi^-\rangle = -|\psi^-\rangle \Rightarrow \langle \psi^- | F | \psi^- \rangle = -1,$$

while for sym. PE state, $F|\chi\rangle = |\chi\rangle \Rightarrow \langle \chi | F | \chi \rangle = +1!$
(i.e.: $|\chi\rangle = |\psi^+\rangle, |\phi^-\rangle, |\psi^+\rangle$)

Mixed states:

$$\rho = \lambda |\psi^-\rangle\langle\psi^-| + (1-\lambda) \frac{\mathbb{1}}{4}, \lambda \in [-\frac{1}{3}, 1]: \text{"Werner state"}$$

$$\text{tr}[F\rho] = \lambda \underbrace{\langle \psi^- | F | \psi^- \rangle}_{=-1} + (1-\lambda) \underbrace{\text{tr}\left(\frac{\mathbb{1}}{4} F\right)}_{=1/2} = \frac{1}{2}(1-3\lambda)$$

\Rightarrow state entangled if $\lambda \geq \frac{1}{3}$.

Optimal? Yes, e.g. $\rho = |0\rangle\langle 0| \otimes |1\rangle\langle 1| \Rightarrow \text{tr}[F\rho] = 0$

(= touches convex set!)

Other witnesses: e.g. $W = \mathbb{1} - d|\mathcal{R}\rangle\langle\mathcal{R}|$; $|\mathcal{R}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i, i\rangle$

\rightarrow Homework!

c) Positive maps and the PPT criterion

(76)

Reminder: A superoperator Λ is called positive if

$$\rho \geq 0 \Rightarrow \Lambda(\rho) \geq 0.$$

Usually we require Λ to be completely positive, i.e.

$$(\Lambda \otimes \mathbb{1})(\rho) \geq 0 \text{ for } \rho \geq 0. \text{ Here, we will however be}$$

interested in positive but not compl. positive maps!

$$\text{Why?} - \text{Consider } \rho_{\text{sep}} = \sum_i p_i \rho_i^A \otimes \rho_i^B.$$

$$\underline{(\Lambda \otimes \mathbb{1})(\rho_{\text{sep}})} = \sum_i p_i \underbrace{\Lambda(\rho_i^A)}_{\tilde{\rho}_i^A} \otimes \rho_i^B = \sum_i p_i \tilde{\rho}_i^A \otimes \rho_i^B \geq \underline{0}$$

$$\text{i.e.: } (\Lambda \otimes \mathbb{1})(\rho) \not\geq 0 \Rightarrow \rho \text{ entangled}$$

↑
"has negative eigenvalues"

Most important example:

$$\Lambda(\rho) := \rho^T \quad (\text{transpose})$$

$$(\Lambda \otimes \mathbb{1})(\rho) \equiv \rho^{TA} \quad : \quad \text{"partial transpose"}$$

$$\text{(i.e.: } \rho = \sum_{ij} \rho_{ij}^{ij} |i\rangle\langle j| \otimes |i\rangle\langle j| \Rightarrow \rho^{TA} = \sum_{ij} \rho_{ij}^{ij} |i'\rangle\langle j'| \otimes |i\rangle\langle j| \text{)}$$

We have thus:

$$\rho^{TA} \neq 0 \Rightarrow \rho \text{ entangled}$$

"PPT (positive partial transpose) criterion"

E.g.: $|\mathcal{R}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i, i\rangle$

$$\Rightarrow (|\mathcal{R}\rangle\langle\mathcal{R}|)^{TA} = \frac{1}{d} \sum (|i, i\rangle\langle j, j|)^{TA} = \frac{1}{d} \sum |j, i\rangle\langle i, j|$$

$\equiv \mathbb{F}$

Not positive: antisym. states have neg. eigenvalue, e.g.

$$|\phi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$$

$$\langle\phi^-| (|\mathcal{R}\rangle\langle\mathcal{R}|)^{TA} |\phi^-\rangle = \frac{1}{2d} (\langle 01| - \langle 10|) (|10\rangle - |01\rangle) = \frac{-1}{2d}$$

Mixed state example:

$$\rho = \lambda |\mathcal{R}\rangle\langle\mathcal{R}| + (1-\lambda) \frac{\mathbb{1}}{d^2}, \quad \frac{-1}{d^2-1} \leq \lambda \leq 1$$

"isotropic state"

E.g. for $d=2$:

$$\rho = \lambda \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+\lambda}{4} & \frac{1-\lambda}{4} & 0 & \frac{1-\lambda}{4} \\ \frac{1-\lambda}{4} & \frac{1-\lambda}{4} & 0 & \frac{1-\lambda}{4} \\ 0 & \frac{1-\lambda}{4} & \frac{1-\lambda}{4} & 0 \\ \frac{1-\lambda}{4} & \frac{1-\lambda}{4} & 0 & \frac{1+\lambda}{4} \end{pmatrix}$$

$\swarrow \frac{\lambda}{2}$
 $\searrow \frac{\lambda}{2}$
 $\swarrow \frac{1-\lambda}{4}$
 $\searrow \frac{1-\lambda}{4}$

$$\Rightarrow \rho^{\text{TA}} = \begin{pmatrix} \frac{1+\lambda}{4} & & & \\ & \frac{1-\lambda}{4} & & \\ & & \frac{1}{2} & \\ & & & \frac{1-\lambda}{4} \end{pmatrix}$$

(78)

\Rightarrow positive if & only if $\frac{1}{2} \leq \frac{1-\lambda}{4} \Rightarrow \lambda \leq \frac{1}{3}$.

Note: Criterion indep. of local unitaries on B \Rightarrow detects all max. ent. states! (\rightarrow stronger than witnesses!)

In fact: PPT criteria detects all entangled states in dimension $d_A \times d_B = 2 \times 2$ and 3×2

(but counterex. exist in 3×3 , 4×2 : "PPT bound ent. states")

Other example: $\Lambda(\rho) = \text{tr}(\rho) \mathbb{1} - \rho$:

$$(\mathbb{1} \otimes \Lambda)(\rho) = \underbrace{\text{tr} \rho}_{=1} \cdot (\mathbb{1} \otimes \text{tr}_A \rho) - \rho = \mathbb{1} \otimes \rho_B - \rho \neq 0$$

$\Rightarrow \rho$ entangled.

"reduction criteria" $\mathbb{1} \otimes \text{tr}_A \rho \neq \rho$ (\rightarrow homework)

d) Relation of witnesses & positive maps

(79)

For each witness W , there is a positive map Λ which detects all states W detects (and in fact more),

interpret witness W as "bipartite state" (better: operator)

map $\Lambda \hat{=}$ Jamiołkowski map of "state" W^T :

$$\Lambda(X) := \text{tr}_A (W^T (X^T \otimes \mathbb{1})) = \text{tr}_A (W (X \otimes \mathbb{1}))^T.$$

(Note: $A \leftrightarrow B$ swapped w.r.t. Chapter I)

$$\text{Then, } \langle \phi | \Lambda(\rho) | \phi \rangle = \langle \phi | \text{tr}_A (W(\rho \otimes \mathbb{1})) | \phi \rangle$$

$$= \text{tr}_A (W(\underbrace{\rho \otimes |\phi\rangle\langle\phi|}_{\text{sep}})) \geq 0 \text{ for } \rho \geq 0,$$

i.e., Λ is a positive map, and

$$\text{tr} [W(A \otimes B)] = \text{tr}_B [\text{tr}_A (W(A \otimes \mathbb{1})) B] = \text{tr} [\Lambda(A)^T B]$$

$$= \sum_{ij} [\Lambda(A)^T]_{ij} B_{ji} = d \langle \Omega | \Lambda(A) \otimes B | \Omega \rangle$$

$$= d \langle \Omega | (\Lambda \otimes \mathbb{1})(A \otimes B) | \Omega \rangle$$

$$\text{Linearity: } \text{tr} (W\rho) = d \langle \Omega | (\Lambda \otimes \mathbb{1})(\rho) | \Omega \rangle$$

i.e.: $\text{tr}(W_p) < 0 \Rightarrow (\Lambda \otimes \mathbb{1})(\rho) \neq 0$

(80)

$\Rightarrow \Lambda$ detects all states which W detects!

e.g.: $W = \mathbb{F}$:

$$\Lambda(X) = \text{tr}_A(\mathbb{F}(X^T \otimes \mathbb{1})) = \mathbb{1} \cdot X^T = X^T$$

\Rightarrow PPT criterion!

Note: PPT strictly stronger: \mathbb{F} could e.g. not detect $|\mathcal{R}\rangle$!

Corollary: A state is separable if and only if

$$(\Lambda \otimes \mathbb{1})(\rho) \geq 0 \text{ for all positive } \Lambda.$$

e) Quantification of mixed state entanglement

How to quantify mixed state entanglement?

i) Entanglement needed to create state, e.g.

min. avg. amount of $E(|\phi\rangle) = S(\text{tr}_B(|\phi\rangle\langle\phi|))$ needed

"entanglement of formation"

$$E_F(\rho) = \min_{\{p_i, |\phi_i\rangle\}} \sum p_i E(|\phi_i\rangle)$$

s.t. $\sum p_i |\phi_i\rangle\langle\phi_i| = \rho$

"entanglement cost" $E_c(\rho) = \lim_{N \rightarrow \infty} \frac{1}{N} E_F(\rho^{\otimes N})$ (81)
 (asympt. cost per copy)

ii) Extractable entanglement:

"Distillable entanglement" $E_D(\rho)$:

$E_D(\rho) = \max$, rate $R = \frac{\Omega}{N}$ achievable w/ LOCC
 protocol E_u s.t. $E_u(\rho^{\otimes N}) \rightarrow |\Omega\rangle\langle\Omega|^{\otimes N}$

Note: $E_F \geq E_c \geq E_D$.

Generally $E_c(\rho) \neq E_D(\rho)$: process not reversible,
 no unique measure.

(E.g.: ρ PPT, $\rho^{TA} \geq 0$. LOCC map preserves PPT

\rightarrow PPT states are undistillable, $E_D(\rho) = 0$.

But for ent. PPT states $E_c(\rho) > 0$. (possible)

Converse question: $\rho^{TA} \neq 0 \Rightarrow E_D(\rho) > 0$ is big

open problem - (existence of "NPT bound entanglement")

Problem: E_F, E_C, E_D very hard to compute!

(82)

→ want computable ent. measures.

Wishlist for ent. measures:

⇒ • LOCC-monotone: cannot be increased by LOCC ⇐

• 0 on sep. states

• additive: $E(\rho \otimes \sigma) = E(\rho) + E(\sigma)$

• continuous: $\rho \approx \sigma \Rightarrow E(\rho) \approx E(\sigma)$ (w/ some $\delta - \epsilon$)

• $E_D \leq E \leq E_C$.

• coincides with $E(|\psi\rangle) = S(\text{tr}_B(|\psi\rangle\langle\psi|))$ on pure states.

(Almost) impossible to get all: LOCC monotonicity most important.

Negativity - a computable ent. measure

Have seen: ρ^{TA} has neg. eigenvalues $\Rightarrow \rho$ entangled.

Use neg. eigenvalues as ent. measure:

$$\text{Negativity } \mathcal{N}(\rho) = \frac{1}{2} \left(\underbrace{\sum_i |\lambda_i(\rho^{TA})|}_{=: \|\rho^{TA}\|_1} - 1 \right)$$

$$= \frac{1}{2} (\|\rho^{TA}\|_1 - 1) = -\sum_{\text{neg. eigenvalues}} \lambda_i(\rho^{TA}), \quad \left(\begin{array}{l} \text{since} \\ \text{tr}(\rho^{TA}) = 1 \end{array} \right)$$

or log-negativity $E_N(\rho) = \log_2 \|\rho^{TA}\|_1$.

83

Properties:

Negativity \mathcal{N} :

- LOCC monotone
- 0 on sep. states, but can be 0 on ent. states
- $\neq E(|\psi\rangle)$ for pure states.
- not additive
- continuous

Log-negativity E_N :

- additive
- 0 on sep., but can be 0 on ent. states
- $\neq E(|\psi\rangle)$ for pure states.
- not an LOCC monotone (!)
- continuous