Problem 4

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$$|\chi\rangle \xrightarrow{p} |\phi^+\rangle$$
$$|\chi\rangle \xrightarrow{1-p} |00\rangle$$

$$\rho_A^{\chi} = \mathrm{Tr}_B |\chi\rangle\langle\chi| = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| \text{ ... eigenvalues } \left(\frac{3}{4}, \frac{1}{4}\right)$$

$$\rho_A^{\phi^+} = \mathrm{Tr}_B |\phi^+\rangle\langle\phi^+| = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \text{ ... eigenvalues } \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\rho_A^{00} = \mathrm{Tr}_B |00\rangle\langle 00| = |0\rangle\langle 0| \qquad \text{ ... eigenvalues } (1, 0)$$

Convertibility criterion:

$$\left(\frac{3}{4}, \frac{1}{4}\right) \prec p\left(\frac{1}{2}, \frac{1}{2}\right) + (1-p)(1,0)$$

$$\frac{p}{2} + (1-p) \geq \frac{3}{4}$$

$$p \leq \frac{1}{2}$$

(2)

3 copies of $|\chi\rangle = |\chi\rangle \otimes |\chi\rangle \otimes |\chi\rangle \doteq$

$$= \sqrt{\tfrac{27}{64}}\,|00\rangle\otimes|00\rangle\otimes|00\rangle + \sqrt{\tfrac{9}{64}}\,|00\rangle\otimes|00\rangle\otimes|11\rangle + \sqrt{\tfrac{9}{64}}\,|00\rangle\otimes|11\rangle\otimes|00\rangle +$$

$$+ \sqrt{\tfrac{9}{64}}\,|11\rangle\otimes|00\rangle\otimes|00\rangle + \sqrt{\tfrac{3}{64}}\,|00\rangle\otimes|11\rangle\otimes|11\rangle + \sqrt{\tfrac{3}{64}}\,|11\rangle\otimes|00\rangle\otimes|11\rangle +$$

$$+ \sqrt{\tfrac{3}{64}}\,|11\rangle\otimes|11\rangle\otimes|00\rangle + \sqrt{\tfrac{1}{64}}\,|11\rangle\otimes|11\rangle\otimes|11\rangle$$

... corresponds to eigenvalues $\left(\frac{27}{64}, \frac{9}{64}, \frac{9}{64}, \frac{9}{64}, \frac{3}{64}, \frac{3}{64}, \frac{3}{64}, \frac{1}{64}\right.$
$= \vec{w}$

3 copies of $|\phi^+\rangle = |\phi^+\rangle\otimes|\phi^+\rangle\otimes|\phi^+\rangle = \sqrt{\tfrac{1}{8}}\,|00\rangle\otimes|00\rangle\otimes|00\rangle + \ldots$

... corresponds to eigenvalues $\left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right) = \vec{v}_3$

2 copies of $|\phi^+\rangle = |\phi^+\rangle\otimes|\phi^+\rangle\otimes|\varphi\rangle = \sqrt{\tfrac{1}{4}}\,|00\rangle\otimes|00\rangle\otimes|00\rangle + \sqrt{\tfrac{1}{4}}\,|00\rangle\otimes|11\rangle\otimes|00\rangle + \ldots$

Some "trash", it's best to choose it unentangled, e.g. $|\varphi\rangle = |00$

... corresponds to eigenvalues $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0\right) = \vec{v}_1$

Trash option $|\varphi\rangle\otimes|\varphi\rangle\otimes|\varphi\rangle = |00\rangle\otimes|00\rangle\otimes|00\rangle$

... corresponds to $(1, 0, 0, 0, 0, 0, 0, 0) = \vec{v}_0$

Convertability condition

$$\vec{w} \prec \underset{\frac{1}{8}}{\underset{\uparrow}{p_3}}\cdot\vec{v}_3 + \underset{\frac{5}{8}}{\underset{\uparrow}{p_2}}\cdot\vec{v}_2 + \underset{\frac{1}{4}}{\underset{\uparrow}{p_0}}\vec{v}_0$$

Yield:

$$P_3 = \frac{3\cdot p_3 + 2\cdot p_2}{3} = \frac{3}{24} + \frac{10}{24} = \underset{\substack{\uparrow \\ \text{Yield for 1 cop}}}{\frac{13}{24}} > \frac{1}{2}$$

Check that this condition is met by checking the majorization inequalities e.g. the first two elements of the vectors

$$\frac{27}{64} + \frac{9}{64} \leq \left(\frac{1}{8}\cdot\frac{1}{8} + \frac{5}{8}\cdot\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8}\cdot\frac{1}{8} + \frac{5}{8}\cdot\frac{1}{4}\right) = \frac{27}{64} + \frac{11}{64}$$

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3 2 copies of $|\chi\rangle$ ...... $\vec{w} = \left(\frac{9}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{16}\right)$

2 copies of $|\phi^+\rangle$ .... $\vec{v}_2 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ ... probability p_2

1 copy of $|\phi^+\rangle$ + trash .... $\vec{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$ ... probability p_1

Trash (something like $|11\rangle \otimes |11\rangle$) ... $\vec{v}_0 = (1, 0, 0, 0)$ ... probability $p_0 = 1 - p_1 - p_2$

$$\vec{w} \prec \sum_{i=0}^{2} p_i \vec{v}_i = \left(p_0 + \frac{p_1}{2} + \frac{p_2}{4}, \frac{p_1}{2} + \frac{p_2}{4}, \frac{p_2}{4}, \frac{p_2}{4}\right)$$

Inequalities:

$$\frac{p_1}{2} + \frac{p_2}{4} + 1 - p_1 - p_2 \geq \frac{9}{16} \Rightarrow \frac{7}{16} \geq \frac{3}{4}p_2 + \frac{1}{2}p_1 \quad (A)$$

$$p_0 + p_1 + \frac{p_2}{2} \geq \frac{11}{16} \Rightarrow \frac{1}{4} \geq \frac{p_2}{2} \quad (B)$$

$$p_0 + p_1 + \frac{3}{4} p_2 \geq \frac{15}{16}$$

$$\frac{1}{16} \geq \frac{1}{4} p_2 \quad C$$

Yield: $\frac{2p_2 + p_1}{2} \overset{A}{\leq} \frac{1}{4} p_2 + \frac{7}{16} \overset{C}{\leq} \frac{8}{16} = \frac{1}{2}$