Lecture "Quantum Information" WS 19/20 — Exercise Sheet #4

Problem 1: Decay of entanglement.
Consider a Bell state $\rho = |\Phi^+\rangle\langle\Phi^+|$, where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Superposition states like ρ generally are not stable, but decay over time. A typical evolution is that the populations (i.e., the diagonal elements) become equal, while the off-diagonal elements decay to zero. Suppose that the state evolves as

$$\rho(t) = p_+|00\rangle\langle 00| + p_-|01\rangle\langle 01| + p_-|10\rangle\langle 10| + p_+|11\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|00\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|11\rangle\langle 00| \ ,$$

with $p_\pm = \frac{1}{4}(1 \pm e^{-t/T_1})$.
For sufficiently long times, this state tends to $\lim_{t\to\infty} \rho(t) = \frac{1}{4}\mathbb{I}$, the maximally mixed state.

1. Write a matrix form of state $\rho(t)$.
2. Take its partial transpose $\rho(t)^{T_B}$ and write its matrix form.
3. Calculate the eigenvalues of $\rho(t)^{T_B}$. (You may use a computer algebra system, though it should not be necessary.)
4. Sketch how the eigenvalues change over time for $T_1 = T_2 = 1$. What it the asymptotic limit?
5. Find time after which the state $\rho(t)$ becomes separable.

Problem 2: Bell inequalities and witnesses.
The CHSH operator

$$C = \vec{n}_1\vec{\sigma} \otimes \vec{n}_0\vec{\sigma} + \vec{n}_1\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} + \vec{n}_3\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} - \vec{n}_3\vec{\sigma} \otimes \vec{n}_0\vec{\sigma}$$

with $\vec{n}_k = (\cos(k\pi/4), 0, \sin(k\pi/4))$ has the property that $|\mathrm{tr}[C\rho]| \le 2$ for all ρ which describe a local hidden variable (LHV) model. Note that any separable state $\rho = \sum p_i \rho_i^A \otimes \rho_i^B$ describes a LHV model.

1. Use C to construct an entanglement witness W. Provide an explicit form of the witness. (You may use that all separable states describe LHV models to prove that $\mathrm{tr}[W\rho] \ge 0$.)
2. In which range of λ does this witness detect Werner states $\rho(\lambda) = \lambda|\Psi^-\rangle\langle\Psi^-| + \frac{1-\lambda}{4}\mathbb{I}$, with $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$? How does it compare to the entanglement witness $W = \mathbb{F}$ discussed in the lecture?

Problem 3: Witnesses and reduction criterion.
Consider $W = \mathbb{I} - d|\Omega\rangle\langle\Omega|$, with $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$.

1. Show that $\mathrm{tr}[W\rho] \ge 0$ for separable states ρ, i.e., W is an entanglement witness.
2. Consider the family
$$\rho_{\mathrm{iso}}(\lambda) = \lambda\frac{\mathbb{I}}{d^2} + (1-\lambda)|\Omega\rangle\langle\Omega|$$
of *isotropic states*. In which range of λ is $\rho_{\mathrm{iso}}(\lambda) \ge 0$? In which range of λ does W detect that $\rho_{\mathrm{iso}}(\lambda)$ is entangled?
3. Consider the case $d = 2$. What does W do on the antisymmetric state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$?
4. Derive the positive map Λ corresponding to the witness W. Prove directly that it is indeed a positive map.
5. In which range of λ does Λ detect that $\rho_{\mathrm{iso}}(\lambda)$ is entangled? What does Λ do on the antisymmetric state $|\Psi^-\rangle$?

Problem 4: Entanglement conversion of multiple copies.

Consider the problem of converting a state $|\chi\rangle = \sqrt{\frac{3}{4}}|00\rangle + \sqrt{\frac{1}{4}}|11\rangle$ to the Bell state $|\Phi^+\rangle = \sqrt{\frac{1}{2}}|00\rangle + \sqrt{\frac{1}{2}}|11\rangle$. As we have seen in the lecture, the maximum success probability for this conversion can be determined using the majorization criterion.

1. Determine the maximum success probability P_1 for converting $|\chi\rangle$ into $|\Phi^+\rangle$.

2. Show that there is a protocol which takes three copies of $|\chi\rangle$ and produces into 3 copies of $|\Phi^+\rangle$ with probability $p_3 = \frac{1}{8}$ and 2 copies with $p_2 = \frac{5}{8}$, respectively. What is the average yield P_3 of $|\Phi^+\rangle$ per copy of $|\chi\rangle$ used?

3. Show that by using 2 copies, the average yield does not improve as compared to one copy.