3. Mixed States

Consider bipart. state $|\psi\rangle_{AB} = \sum c_{ij} |i\rangle|j\rangle$

We have only access to A.

$\rightarrow$ How can we characterize measurement on A?

Meas. M on A $\Longleftrightarrow$ meas. $M_A \otimes \mathbb{1}_B$ on A+B.

$$\langle\psi|M_A \otimes \mathbb{1}_B|\psi\rangle = \sum c_{i'j'}^* \langle i'|\langle j'|(M_A \otimes \mathbb{1}_B)|i\rangle|j\rangle c_{ij}$$

$$= \sum c_{i'j'}^* c_{ij} \langle i'|M_A|i\rangle \underbrace{\langle j'|j\rangle}_{=\delta_{jj'}}$$

$$= \sum_{ii'} \Big(\sum_j c_{i'j}^* c_{ij}\Big) \langle i'|M_A|i\rangle = (*)$$

Define ρ_A ($d_A \times d_A$ matrix) via $(\rho_A)_{ii'} = \sum_j c_{i'j}^* c_{ij} = CC^\dagger$

(with $C = (c_{ij})_{ij}$), or equiv. $\rho_A = \sum_{ii'j} c_{i'j}^* c_{ij} |i\rangle\langle i'|$

... $(*) = \mathrm{tr}[\rho_A M]$,

with the <u>trace</u> $\mathrm{tr}(X) = \sum \langle k|X|k\rangle$. ONB!

<u>Note</u>: The trace is cyclic: $\mathrm{tr}(AB) = \sum_k \langle k|AB|k\rangle$

$$= \sum_{kl} \langle k|A|l\rangle\langle l|B|k\rangle = \sum_{kl} \langle l|B|k\rangle\langle k|A|l\rangle$$

$$= \sum \langle l|BA|l\rangle = \mathrm{tr}(BA).$$

and thus basis-indep.: $tr(X) = tr(U^\dagger U X) = tr(U X U^\dagger)$. (19)

ρ_A is called <u>density operator</u> or <u>density matrix</u>, or <u>mixed state</u>.
It characterizes systems where we only have <u>partial knowledge</u>.

<u>Properties of ρ_A:</u>

- $\rho_A = CC^\dagger \Rightarrow \rho_A^\dagger = (CC^\dagger)^\dagger = CC^\dagger = \rho_A$
- ρ_A is positive semi-definite (= all eigenvalues ≥ 0), $\rho_A \geq 0$

$$\langle\phi|\rho_A|\phi\rangle = \langle\phi|CC^\dagger|\phi\rangle = (C^\dagger|\phi\rangle)^\dagger(C^\dagger|\phi\rangle) \geq 0 \quad \forall|\phi\rangle.$$

- $tr(\rho_A) = \sum_i (CC^\dagger)_{ii} = \sum_{ij} c_{ij} c_{ij}^* = \langle\psi|\psi\rangle = 1.$

<u>Properties of density operators:</u>

- $\rho_A^\dagger = \rho_A$
- $\rho_A \geq 0$
- $tr(\rho_A) = 1$

<u>Note:</u> Consequence: For $0<p<1$, ρ, ρ' density ops, $p\rho + (1-p)\rho'$ is also density op $\Rightarrow$ density ops form <u>convex set</u>!

Is ρ_A uniquely determined by

$$\mathrm{tr}[M\rho_A] = \langle\psi|_{AB}\; M\otimes\mathbb{1}\;|\psi\rangle_{AB}\;?$$

<u>Yes</u>: $\mathrm{tr}[X^\dagger Y]$ is scalar product, & overlap of ρ_A w/ all herm. M determines herm. part of ρ_A <u>entirely</u>!

(<u>Consequence</u>: All numbers in ρ_A meaningful $\Rightarrow$ no phase ambiguity!)

What <u>is</u> ρ_A for pure state $|\phi\rangle_A$?

$$\langle\phi|M|\phi\rangle = \mathrm{tr}[\langle\phi|M|\phi\rangle] = \mathrm{tr}[M\,|\phi\rangle\langle\phi|]$$

$\Rightarrow$ $\rho = |\phi\rangle\langle\phi|$ (projector onto $|\phi\rangle$).

<u>Partial trace</u>:

Given <u>general</u> state ρ_{AB} on $A+B$ (e.g. $\rho_{AB}=|\psi\rangle\langle\psi|$), what is descr. of meas M_A on A?

$$\mathrm{tr}[(M\otimes I)\,\rho_{AB}] = \sum_{ij\,i'j'} \langle i\underbrace{j|M\otimes I|i'j'}_{\delta_{jj'}}\rangle\langle i'j'|\rho_{AB}|ij\rangle$$

$$= \sum \langle i|M|i'\rangle\langle i'j|\rho_{AB}|ij\rangle = \mathrm{tr}[M\cdot\rho_A],$$

Is ρ_A uniquely determined by

$$\mathrm{tr}[M\rho_A] = \langle\psi|_{AB}\; M\otimes 1\; |\psi\rangle_{AB}\ ?$$

<u>Yes</u>: $\mathrm{tr}[X^\dagger Y]$ is scalar product, & overlap of ρ_A w/ all herm. M determines herm. part of ρ_A <u>entirely</u>!

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$$\mathrm{tr}\big[(M\otimes I)\,\rho_{AB}\big] = \sum_{ij\,i'j'} \langle ij|M\otimes I|i'j'\rangle\langle i'j'|\rho_{AB}|ij\rangle$$

$\delta_{jj'}$

$$= \sum \langle i|M|i'\rangle\langle i'j|\rho_{AB}|ij\rangle = \mathrm{tr}\big[M\cdot\rho_A\big],$$

Where we def. the <u>partial trace</u>

$$\rho_A = \sum |i'\rangle\langle i' j| \rho_{AB} |ij\rangle\langle i|$$

$$= \sum \left(\mathbb{1}_A \otimes \langle j'|_B\right)\left(\rho_{AB}\right)\left(\mathbb{1}_A \otimes |j\rangle_B\right)$$

$$= \sum_j \langle j|_B \, \rho_{AB} \, |j\rangle_B$$

$$=: \underline{\underline{\mathrm{tr}_B \, \rho_{AB}}}$$

(In components: $\left(\mathrm{tr}_B \, \rho_{AB}\right)_{ii'} = \sum_j \left(\rho_{AB}\right)_{(ij)(i'j)}$)

Is any density matrix ρ <u>physical</u>?

Take $\rho = \sum \lambda_i |\phi_i\rangle\langle\phi_i|$ eigenval. decomp.; and

def $|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\, |\phi_i\rangle_A |i\rangle_B$ ("purification" of ρ)

$$\Rightarrow \mathrm{tr}_B\left[|\psi\rangle_{AB}\langle\psi|_{AB}\right] = \mathrm{tr}_B\left[\sum \sqrt{\lambda_i \lambda_j}\, |\phi_i\rangle\langle\phi_j| \otimes |i\rangle\langle j|\right]$$

$$= \sum \lambda_i |\phi_i\rangle\langle\phi_i| = \rho \quad \Rightarrow \underline{\underline{\text{yes}}} \checkmark$$

Density matrix can serve as <u>alternative definition</u> of state.

Ensemble interpretation of density matrix

Consider $|\psi\rangle_{AB} = \alpha|00\rangle + \beta|11\rangle$

$$\Rightarrow \rho_A = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} = |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 \, |1\rangle\langle 1|$$

$$\mathrm{tr}[\Pi \rho_A] = |\alpha|^2 \, \mathrm{tr}[\Pi |0\rangle\langle 0|] + |\beta|^2 \, \mathrm{tr}[\Pi |1\rangle\langle 1|]$$

$\Rightarrow$ Can be interpreted as having $|0\rangle$ w/ prob. $p_0 = |\alpha|^2$ & $|1\rangle$ w/ prob. $p_1 = |\beta|^2$. "<u>ensemble interpretation</u>"

$\Rightarrow$ Is this consistent w/ <u>pure</u> state $|\psi\rangle_{AB}$?

Let B do proj. meas. in Z basis:

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle \quad \begin{array}{l} \xrightarrow{p_0 = |\alpha|^2} \; |\psi_0\rangle_A = |0\rangle_A \\ \quad Z \text{ meas. on } B \\ \xrightarrow[p_1 = |\beta|^2]{} \; |\psi_1\rangle_A = |1\rangle_A \end{array}$$

$\Rightarrow$ Alice doesn't know outcome $\Rightarrow$ <u>ensemble</u>

$$\{(p_0; |0\rangle), (p_1; |1\rangle)\} \equiv \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}.$$

(<u>Note</u>: Bob's description is different: he knows outcome and would describe the state as <u>either</u> $|0\rangle\langle 0|$ <u>or</u> $|1\rangle\langle 1|$)

But: Bob could also meas. in $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

basis!

$$p_+ = \frac{|\alpha|^2 + |\beta|^2}{2} = \frac{1}{2} \qquad |\psi_+\rangle_A = \frac{\alpha|0\rangle + \beta|1\rangle}{|\alpha|^2 + |\beta|^2}$$

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle \quad X \text{ meas.}$$

not orthogonal!

$$p_- = \frac{|\alpha|^2 + |\beta|^2}{2} = \frac{1}{2} \qquad |\psi_-\rangle_A = \frac{\alpha|0\rangle - \beta|1\rangle}{|\alpha|^2 + |\beta|^2}$$

Different ensemble $\quad \rho_A = p_+ |\psi_+\rangle\langle\psi_+| + p_- |\psi_-\rangle\langle\psi_-|$

for same state $\Longrightarrow$ ens. interpretation ambiguous!

(Number of terms can vary ($\rightarrow$ HW!), non-orth. states as $|\psi_\pm\rangle$, ...)

How are diff. ensembles related?

Note: Not orthogonal, just $\langle\psi_i|\psi_i\rangle = 1$

Theorem: Let $\rho = \sum p_i |\psi_i\rangle\langle\psi_i| = \sum q_j |\phi_j\rangle\langle\phi_j|$.

Then, there exists a unitary $U = (u_{ij})$ s.th.

$$\sqrt{p_i}\, |\psi_i\rangle = \sum_j u_{ij} \sqrt{q_j}\, |\phi_j\rangle ,$$

and vice versa. (If there are less j's than i's, pad with zeros, and vice versa.)

Proof: "$\Leftarrow$": Let $\sqrt{p_i}\,|\psi_i\rangle = \sum_j u_{ij}\sqrt{q_j}\,|\phi_j\rangle$.

Then
$$\sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i \Big(\sum_j u_{ij}\sqrt{q_j}\,|\phi_j\rangle\Big)\Big(\sum_{j'} u^*_{ij'}\sqrt{q_{j'}}\,\langle\phi_{j'}|\Big)$$
$$= \sum_{jj'} \sqrt{q_j q_{j'}}\,|\phi_j\rangle\langle\phi_{j'}| \underbrace{\Big(\sum_i u^*_{ij'} u_{ij}\Big)}_{=\delta_{jj'}}$$
$$= \sum q_j |\phi_j\rangle\langle\phi_j| .$$

"$\Rightarrow$": Homework / see later (equiv. of purifications).

4. Schmidt decomposition and purifications

Given $|\psi\rangle_{AB}$ bipartite, let

$$\mathrm{tr}_B |\psi\rangle\langle\psi| = \rho_A = \sum p_i |i\rangle_A \langle i|_A$$

with $|i\rangle_A$ eigenbasis (ONB) ("abuse" of notation...)

Choose <u>some</u> ONB $|a_j\rangle_B$ of B, expand

$$|\psi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A |a_j\rangle_B$$
$$= \sum_i |i\rangle_A \Big(\sum_j c_{ij} |a_j\rangle_B\Big) =: |b_i\rangle :$$

<u>no ONB</u>