Proof: "$\Leftarrow$": Let $\sqrt{p_i}\,|\psi_i\rangle = \sum_j u_{ij}\sqrt{q_j}\,|\phi_j\rangle$.

Then $$\sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i \Big(\sum_j u_{ij}\sqrt{q_j}\,|\phi_j\rangle\Big)\Big(\sum_{j'} u^*_{ij'}\sqrt{q_{j'}}\,\langle\phi_{j'}|\Big)$$

$$= \sum_{jj'} \sqrt{q_j q_{j'}}\,|\phi_j\rangle\langle\phi_{j'}| \underbrace{\Big(\sum_i u^*_{ij'} u_{ij}\Big)}_{=\delta_{jj'}}$$

$$= \sum q_j |\phi_j\rangle\langle\phi_j| .$$

"$\Rightarrow$": Homework / see later (equiv. of purifications).

4. Schmidt decomposition and purifications

Given $|\psi\rangle_{AB}$ bipartite, let

$$\mathrm{tr}_B |\psi\rangle\langle\psi| = \rho_A = \sum p_i |i\rangle_A \langle i|_A$$

with $|i\rangle_A$ eigenbasis (ONB) ("abuse" of notation...)

Choose some ONB $|a_j\rangle_B$ of B, expand

$$|\psi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A |a_j\rangle_B$$

$$= \sum_i |i\rangle_A \Big(\sum_j c_{ij} |a_j\rangle_B\Big) =: |\tilde{b}_i\rangle : \text{ no ONB}$$

$$\ldots = \sum |i\rangle_A |b_i\rangle_B$$

We have $\sum_i p_i |i\rangle\langle i| = \mathrm{tr}_B |\psi\rangle\langle\psi| = \mathrm{tr}_B \left(\sum_{ii'} |i\rangle\langle i'|_A \otimes |b_i\rangle\langle b_{i'}|_B \right)$

$$= \sum_{ii'} |i\rangle\langle i'|_A \otimes \mathrm{tr}\left(|b_i\rangle\langle b_{i'}|_B \right)$$

$$= \sum_{ii'} \langle b_{i'} | b_i \rangle \cdot |i\rangle\langle i'|$$

Since $|i\rangle\langle i'|$ is basis (lin. indep.) in space of matrices:

$$\Rightarrow \langle b_{i'} | b_i \rangle = \delta_{i'i}\, p_i$$

$$\Rightarrow |i\rangle_B := \frac{1}{\sqrt{p_i}} |b_i\rangle \text{ is } \underline{\text{ONB for B}}$$

different basis than $|i\rangle_A$ ($\rightarrow$ notation!)

<u>Schmidt decomposition</u>:

Any $|\psi\rangle_{AB}$ can be written as

$$|\psi\rangle_{AB} = \sum_i \lambda_i |i\rangle_A |i\rangle_B ,$$

with ONBs $|i\rangle_A$ & $|i\rangle_B$. The $\lambda_i = \sqrt{p_i} \geq 0$ are called Schmidt coefficients.

Note: $\rho_B = \mathrm{tr}_A |\psi\rangle\langle\psi| = \sum_i \lambda_i^2 |i\rangle_B \langle i|_B$

$\Rightarrow |i\rangle_B$ eigenbasis of ρ_B!

$\Rightarrow$ If p_i non-degen.: Schmidt decomp. obtained by pairing eigenvectors of ρ_A & ρ_B.

Important consequence: For pure states $|\psi\rangle_{AB}$, ρ_A and ρ_B have the same eigenvalues!

How is Schmidt dec. related to other expansions?

$$|\psi\rangle = \sum c_{ij} |x_i\rangle_A |y_j\rangle_B$$

some ONBs

$$= \sum \lambda_k |k\rangle_A |k\rangle_B .$$

$|x_i\rangle_A, |y_j\rangle_B, |k\rangle_A, |k\rangle_B$ ONBs

$\Rightarrow \exists$ unitaries u_{ik}, v_{jk} s.th.

$$|k\rangle_A = \sum u_{ik} |x_i\rangle_A \; ; \; |k\rangle_B = \sum v_{jk}^* |y_j\rangle_B$$

(pad w/ zeros if necessary...)

$$\Rightarrow \sum c_{ij} \, |x_i\rangle_A |y_j\rangle_B = \sum \lambda_k u_{ik} v_{jk}^* \, |x_i\rangle_A |y_j\rangle_B$$

$\xrightarrow{\text{lin. indep. of } |x_i\rangle_A |y_j\rangle_B}$

$$c_{ij} = \sum_k u_{ik} \lambda_k v_{jk}^* ,$$

or $\quad C = U \cdot D \cdot V^\dagger \qquad \left(C \equiv (c_{ij}) \right)$

with U, V unitary, and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \\ & & & 0 \end{pmatrix}$

$\boxed{\text{"Singular value decomposition" (SVD) of } C}$

(Derivation of SVD ($\rightarrow$ HW): U diagonalizes $CC^\dagger$, V $C^\dagger C$
$\Leftrightarrow$ derivation of Schmidt decomp.!)

<u>Remark</u>: Any two states $|\phi\rangle, |\psi\rangle$ w/ ident. Schmidt coeffs are related by local unitaries, i.e.

$$\exists U, V : \ |\phi\rangle = U \otimes V |\psi\rangle .$$

$\Rightarrow$ the λ_i contain <u>all</u> non-local properties,

$\lambda_1 \geq \lambda_2 \geq \ldots$

Proof: $|\phi\rangle = \sum \lambda_i |\phi_i^A\rangle \otimes |\phi_i^B\rangle$ (ONBs)

$|\psi\rangle = \sum \lambda_i |\psi_i^A\rangle \otimes |\psi_i^A\rangle$ (ONBs)

$|\phi_i^A\rangle, |\psi_i^A\rangle$ ONB $\Rightarrow \exists U : |\phi_i^A\rangle = U|\psi_i^A\rangle \ \forall i$

& same for B : $\exists V : |\phi_i^B\rangle = V|\psi_i^B\rangle \ \forall i$ ∎

(Again: Pad w/ 0 if necessary.)

<u>Purification</u>:

Any $|\psi\rangle_{AB}$ s.th. $\mathrm{tr}_B |\psi\rangle\langle\psi|_{AB} = \rho_A$ is called a <u>purification</u> of ρ_A.

need not be orthogonal!

(E.g. $\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i| \Rightarrow \sum \sqrt{p_i}|\psi_i\rangle|i\rangle$ is purif.)

Given two purifications $|\phi\rangle$ & $|\psi\rangle$ of ρ_A, what is their relation?

Write $|\phi\rangle, |\psi\rangle$ in <u>Schmidt form</u>:

$$|\phi\rangle = \sum \lambda_i |\phi_i^A\rangle |\phi_i^B\rangle$$

(all ONBs)

$$|\psi\rangle = \sum \mu_i |\psi_i^A\rangle |\psi_i^B\rangle$$

λ_i, μ_i wlog descending.

We have

$$\sum \lambda_i |\phi_i^A\rangle\langle\phi_i^A| = \mathrm{tr}_B |\phi\rangle\langle\phi| = \mathrm{tr}_B |\psi\rangle\langle\psi| = \sum \mu_i |\psi_i^A\rangle\langle\psi_i^A|$$

$\Rightarrow \lambda_i = \mu_i$, $|\phi_i^A\rangle = |\psi_i^A\rangle$ (up to phase)

if λ_i non-degen. (degen. $\rightarrow$ HW)

Now choose U s.th. $U|\phi_i^B\rangle = |\psi_i^B\rangle$ $\forall i$ ($\Rightarrow U$ unitary)

$$\Rightarrow \quad |\psi\rangle = (\mathbb{1} \otimes U)|\phi\rangle.$$

All purifications are related by a unitary on the purifying system.

(Note: Closely related to unitary equivalence of ensemble decompositions $\rightarrow$ HW!)

36. Mixed states – unitary evolution + projective measurement

Unitary evolution of mixed state

How does a mixed state ρ_A evolve under a unitary U_A?

Consider purification $|\psi\rangle_{AB}$, $\mathrm{tr}_B|\psi\rangle\langle\psi| = \rho_A$.

$$|\psi\rangle \longmapsto (U_A \otimes \mathbb{1}_B)|\psi\rangle$$

$$\Rightarrow \rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi| \longmapsto \mathrm{tr}_B\left[(U_A \otimes \mathbb{1}_B)|\psi\rangle\langle\psi|(U_A^\dagger \otimes \mathbb{1}_B)\right]$$

$$= U_A \cdot \mathrm{tr}_B\left[(\mathbb{1}_A \otimes \mathbb{1}_B)|\psi\rangle\langle\psi|(\mathbb{1}_A \otimes \mathbb{1}_B)\right] U_A^\dagger$$

$$= \underline{\underline{U_A \rho_A U_A^\dagger}}$$

(Alt. derivation: $\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i|$ & $|\psi_i\rangle \mapsto U_A|\psi_i\rangle$)

<u>Measurement of mixed states:</u>

Proj. measurement E_n:

Have seen: $p_n = \mathrm{tr}[E_n \rho_A]$.

Post-meas. state:

$$\rho_{A,n} = \frac{1}{p_n} \mathrm{tr}_B\left[(E_n \otimes \mathbb{1})|\psi\rangle\langle\psi|(E_n^\dagger \otimes \mathbb{1})\right]$$

$$= E_n \rho_A E_n^\dagger .$$

5. POVM measurements

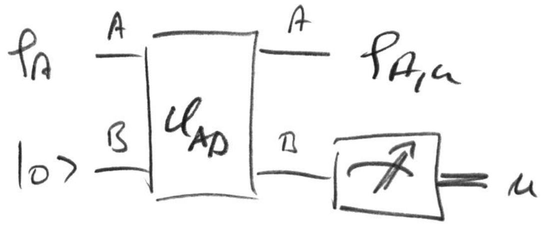
Have seen: add'l system B → more rich situation

What <u>measurements</u> can we realize by adding extra system?

<u>Idea:</u> i) Add "ancilla" B in state $|0\rangle$

ii) act w/ unitary U_{AB}

iii) measure B in $|0\rangle, \dots, |d_B - 1\rangle$



Post-meas. state (un-normalized):

$$\tilde{\rho}_n^A = \langle n|_B U \left(\rho_A \otimes |0\rangle\langle 0|_B \right) U^\dagger |n\rangle_B$$

$$= M_n \rho_A M_n^\dagger \ , \text{ with } M_n := \langle n|_B U |0\rangle_B$$

$$\equiv \left(\mathbb{1}_A \otimes \langle n|_B \right) U \left(\mathbb{1}_A \otimes |0\rangle_B \right)$$

and $\underline{\underline{p_n}} = \mathrm{tr}\, \tilde{\rho}_n^A = \mathrm{tr}\left(M_n \rho_A M_n^\dagger \right) = \underline{\mathrm{tr}\left(M_n^\dagger M_n \rho_A \right)}$.

We have

$$\underline{\underline{\sum M_n^\dagger M_n}} = \sum_n \langle 0|_B U^\dagger |n\rangle_B \langle n|_B U |0\rangle_B = \langle 0|_B \mathbb{1}_{AB} |0\rangle_B = \underline{\underline{\mathbb{1}_A}}$$

(Ensures $\sum p_u = \sum tr(\Pi_u^\dagger \Pi_u \rho_A) = tr(\rho_A) = 1$)

Definition: A set $\{F_u = \Pi_u^\dagger \Pi_u\}$ with $0 \leq F_u \leq \mathbb{1}$, $\sum F_u = \mathbb{1}$, is called <u>positive operator-valued measure</u>, and the corresp. measurement w/ outcome probs $p_u = tr[\Pi_u^\dagger \Pi_u \rho] = tr[F_u \rho]$ a <u>POVM measurement</u>.

Can any $\{\Pi_u\}$ w/ $\sum \Pi_u^\dagger \Pi_u = \mathbb{1}$ be realized by extensions + unitaries?

$$\begin{pmatrix} \Pi_0 \\ \vdots \\ \Pi_{d-1} \end{pmatrix} \quad \sum \Pi_u^\dagger \Pi_u = \mathbb{1} : \text{orth. columns} \quad \Longrightarrow \quad \text{can be extended to unitary}$$

$$U = \begin{array}{c|cc} & |0\rangle_B & |1\rangle_B \; \cdots \\ \hline \langle 0|_B & \Pi_0 & \\ \langle 1|_B & \Pi_1 & \\ \vdots & \vdots & \\ \langle d-1|_B & \Pi_{d-1} & \end{array}$$

i.e.: $\langle u|_B U |0\rangle_B = \Pi_u$.

$\Rightarrow$ any POVM meas. $\{\Pi_u\}$ can be realized by unitary U + projective measurement!

Is this the most general measurement?

Most gen. linear model: Set $\{F_n\}$ s.th. $p_n = \mathrm{tr}[F_n \rho]$.

Wlog., we can choose $F_n = F_n^\dagger$. If not, write

$$F_n = \underbrace{\tfrac{1}{2}(F_n + F_n^\dagger)}_{\text{herm. part}} + \underbrace{\tfrac{1}{2}(F_n - F_n^\dagger)}_{\text{anti-herm. part}}$$

$$\mathrm{tr}[(F_n - F_n)^\dagger \rho] \underset{\substack{\uparrow \\ p_n \geq 0:\ \text{trace } \underline{\text{real}}!}}{=} \mathrm{tr}[((F_n - F_n^\dagger)\rho)^\dagger] = \mathrm{tr}[\rho \cdot (F_n^\dagger - F_n)]$$
$$= -\mathrm{tr}[(F_n - F_n^\dagger)\rho]$$

$\Rightarrow \mathrm{tr}[(F_n - F_n)^\dagger \rho] = 0 \Rightarrow$ <u>assume F_n hermitian!</u>

<u>Conditions:</u>

- $1 = \sum p_n = \mathrm{tr}[(\sum F_n)\rho]$ for all $\rho \Rightarrow \underline{\underline{\sum F_n = \mathbb{1}}}$.
- $0 \leq p_n = \mathrm{tr}[F_n \rho]. \Rightarrow F_n \geq 0$

  (otherwise choose $|\phi\rangle$ s.th. $F_n|\phi\rangle = \lambda|\phi\rangle$, $\lambda < 0$:

  $$\mathrm{tr}[F_n |\phi\rangle\langle\phi|] = \lambda < 0 \quad \lightning \;)$$

Moreover, any $F_n \geq 0$ is of the form $F_n = M_n^\dagger M_n$, e.g.

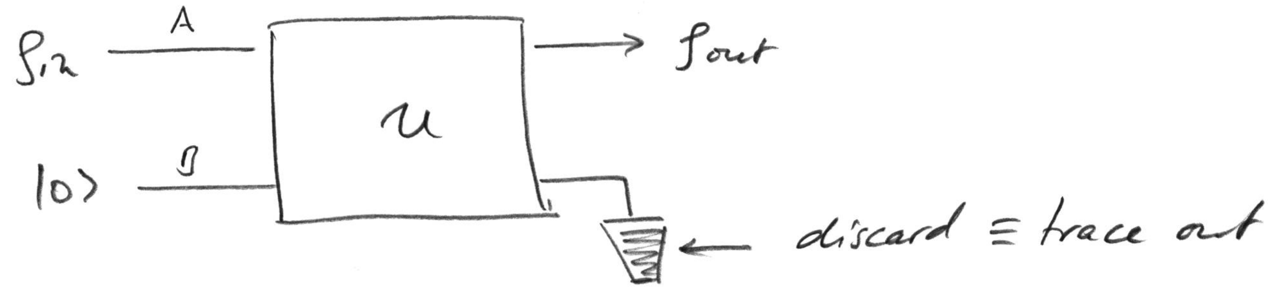
$$F_n = \sum \lambda_i |\phi_i\rangle\langle\phi_i| \Rightarrow M_n = \begin{pmatrix} \sqrt{\lambda_1}\langle\phi_1| \\ \sqrt{\lambda_2}\langle\phi_2| \\ \vdots \end{pmatrix}$$

$\Rightarrow$ Most general measurement!

6. General evolution – superoperators

Q.: What is the most general physical map on density matrices ("superoperator")?

Idea: Try to add ancilla:



$$\rho \mapsto \mathcal{E}(\rho) = \mathrm{tr}_B\left[U(\rho\otimes|0\rangle\langle 0|)U^\dagger\right]$$

$$= \sum_n \underbrace{\langle n|_B U|0\rangle_B}_{=:\,M_n} \rho \langle 0|_B U^\dagger|n\rangle_B$$

$$= \sum_n M_n \rho M_n^\dagger .$$

(Note: trace in diff. basis $\Rightarrow$ diff. M_n: not unique!)

Properties of M_n? As before:

$$\sum M_n^\dagger M_n = \sum \langle 0|_B U|n\rangle_B \langle n|_B U^\dagger|0\rangle_B = \mathbb{1}_A .$$

Kraus representation:

We call

$$\mathcal{E}(\rho) = \sum M_n \rho M_n^\dagger \;; \quad \sum M_n^\dagger M_n = \mathbb{1}$$

the <u>Kraus form</u> or <u>Kraus representation</u> of $\mathcal{E}$.

<u>Note</u>: Any such $\mathcal{E}$ can be realized by ancilla + unitary (cf. POVM). In fact, $\mathcal{E}$ can be seen as POVM where we ignore result (meas. by environment).

Is this the most general physical evolution?

<u>Conditions for physical evolution $\mathcal{E}$:</u>

(i) hermiticity-preserving: $\rho = \rho^\dagger \Rightarrow \mathcal{E}(\rho) = \mathcal{E}(\rho)^\dagger$

(ii) positive: $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) \geq 0$.

(iii) trace-preserving: $\mathrm{tr}(\rho) = 1 \Rightarrow \mathrm{tr}(\mathcal{E}(\rho)) = 1$

(iv) linear: $\mathcal{E}(\rho + \lambda\sigma) = \mathcal{E}(\rho) + \lambda\,\mathcal{E}(\sigma)$

(Note: w/out linearity, ensemble interpretation breaks down $\rightarrow$ HW)